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ORIGINAL PAPER

A Newton-like method for generalized operator equations in Banach spaces

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Abstract In this paper, we are concerned with the semilocal convergence analysis of a Newton-like method discussed by Bartle (Amer Math Soc **6**: 827–831, 1955) to solve the generalized operator equations containing nondifferentiatble term in Banach spaces. This method has also been studied by Rheinboldt (SIAM J Numer Anal **5**: 42–63, 1968). The aim of the paper is to discuss the convergence analysis under local Lipschitz condition $\|F'_x - F'_{x_0}\| \leq \omega(\|x - x_0\|)$ for a given point x_0. Our results extend and improve the previous ones in the sense of local Lipschitz conditions. We apply our results to solve the Fredholm-type operator equations.

Keywords Banach space · Generalized operator equation · Fréchet derivative · Newton-like method · Semilocal convergence · Nonlinear Fredholm-type operator equation

Mathematics Subject Classifications (2010) 65J15 · 49M15

1 Introduction

Let X and Y be two Banach spaces and D be an open convex subset of X. Throughout the paper, we denote by $B_r[x]$ the closed ball defined as $B_r[x] = \{y \in X : \|y -$

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$x\| \leq r\}$, $B(X, Y)$ the Banach space of bounded linear operators from X to Y, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and Φ denotes the collection of all nondecreasing and continuous functions $\phi : [0, \infty) \to [0, \infty)$. Consider the nonlinear operator equation

$$F(x) = 0, \tag{1.1}$$

where F is an operator defined from D into Y such that F is continuously Fréchet differentiable at each point of D. The problems of differential and integral equations, differential inequalities, optimization problems, variational problems, fixed points and many others can be formulated in terms of finding the solution of nonlinear operator equations (1.1) (see [10, 12, 13, 15, 18, 31]).

In his work Bartle [11] used the following iteration process for the solving of operator (1.1):

$$x_{n+1} = x_n - A_n{}^{-1}F(x_n), \text{ for all } n \in \mathbb{N}_0, \tag{1.2}$$

where $\{A_n\}_{n=0}^{\infty}$ is a sequence of bounded linear operators from X into Y satisfying the conditions

$$\left\| A_n - F'_{x_0} \right\| < \frac{1}{4\lambda} \text{ and } \left\| A_n^{-1} \right\| < \lambda,$$

for some $\lambda > 0$ (see [26]). In [19], Pereyra discussed the convergence analysis of (1.2) for solving the least square problems. In [23], Rheinboldt discussed the convergence analysis of (1.2) for solving (1.1) using the Lipschitz condition:

$$\left\| F'_x - F'_y \right\| \leq K\|x - y\|, \text{ for all } x, y \in D \text{ and for some } K > 0. \tag{1.3}$$

In [1, 14, 18, 20], the following general condition is considered for convergence of Newton-like method:

$$\left\| F'_x - F'_y \right\| \leq \omega(\|x - y\|), \tag{1.4}$$

where ω is a given real-valued function.

It is interesting to consider the following condition:

$$\left\| F'_x - F'_{x_0} \right\| \leq \omega(\|x - x_0\|). \tag{1.5}$$

It is easy to see that (1.5) is a weaker assumption than (1.4) (see also [17]).

In this paper, we consider the problem

$$F(x) + G(x) = 0, \tag{1.6}$$

where F and G are two operators defined on an open convex subset D of a Banach spaces X into another Banach space Y such that F is continuously Fréchet differentiable at each point of D and G is not necessarily differentiable. The operator equation (1.6) is discussed in [3–9, 22, 24, 25, 27–29] and references therein.

Motivated by Bartle [11], the purpose of this paper is to introduce a Newton-like method for solving (1.6) and discuss its semilocal convergence analysis under the local Lipschitz condition (1.5). Since our assumptions on nonlinear operators

F and G involving in operator equation (1.6) are fairly general, our main result (Theorem 3.1) covers a wide variety of nonlinear operator equations and it significantly improves the corresponding results of Argyros [2], Bartle [11] and Rheinboldt [23]. As an example, we apply our results to the class of Fredholm-type operator equations.

2 Preliminaries

In subsequent sections, we shall make use of the following lemmas.

Lemma 2.1 (Rall [21], Page 50) *Let L be a bounded linear operator on a Banach space X. Then the followings are equivalent:*

(a) *There is a bounded linear operator M on X such that M^{-1} exists, and*

$$\|M-L\| < \frac{1}{\|M^{-1}\|}.$$

(b) *L^{-1} exists.*

Further, if L^{-1} exists, then $\|L^{-1}\| \leq \frac{\|M^{-1}\|}{1-\|1-M^{-1}L\|} \leq \frac{\|M^{-1}\|}{1-\|M^{-1}\|\|M-L\|}$.

Lemma 2.2 *Let a, b and c be three positive real numbers and let $\phi \in \Phi$. Suppose that the function $\varphi : [0,\infty) \to (-\infty,\infty)$ defined by*

$$\varphi(s) = \int_0^s a(\phi(t)+b)dt - s + c \text{ for all } s \in [0,\infty) \tag{2.1}$$

has a minimum positive zero r^*. Then the sequence $\{t_n\}$ defined by

$$t_0 = 0, t_{n+1} = t_n + \varphi(t_n), n \in \mathbb{N}_0 \tag{2.2}$$

is nondecreasing and converges to r^*. Further, if there exists a point $r^{**} \in (0,\infty)$ such that $\varphi(r^{**}) \leq 0$ and r^* is the unique zero of $\varphi(r)$ in $(0, r^{**}]$, the sequence $\{s_n\}$ defined by

$$s_0 = r^{**}, s_{n+1} = s_n + \varphi(s_n), n \in \mathbb{N}_0$$

is nonincreasing and hence converges to r^*.

Proof First, we show that the sequence $\{t_n\}$ is bounded above by r^*. Since $r^* > 0$ and $t_0 = 0$, we have $t_0 < r^*$. Note that

$$r^* - c = \int_0^{r^*} a(\phi(t)+b)dt > 0.$$

Hence $t_1 = c < r^*$. Suppose that $t_n < r^*$ for some integer $n \geq 1$. By the definition of $\{t_n\}$, we obtain

$$r^* - t_{n+1} = r^* - \int_0^{t_n} a(\phi(t)+b)dt - c = \int_{t_n}^{r^*} a(\phi(t)+b)dt > 0,$$

which implies that

$$t_{n+1} < r^*.$$

Thus, by induction, $t_n < r^*$ for all $n \in \mathbb{N}_0$ and hence $\{t_n\}$ is bounded.

Next, we show that $\{t_n\}$ is a monotonically increasing. Since r^* is minimum positive zero of $\varphi(s)$, we have $\varphi(s) > 0$ for all $s \in (0, r^*)$. Since $t_n < r^*$, we have $\varphi(t_n) > 0$ for all $n \in \mathbb{N}_0$. Hence, we conclude that $\{t_n\}$ is a nondecreasing sequence. Therefore, $\{t_n\}$ is a convergent sequence and converges to t^*, for some $t^* \in [c, r^*]$. From (2.1), φ is continuous, it follows from (2.2) that $t^* = t^* + \varphi(t^*)$, i.e., $\varphi(t^*) = 0$. Note that r^* is the unique zero of φ. Therefore $t^* = r^*$. It is easy to see that the sequence $\{s_n\}$ is nonincreasing, bounded and converges to r^*. □

3 Newton-like algorithm and its convergence analysis

Let F and G be two operators defined on an open convex subset D of a Banach space X with values in a Banach space Y such that F is Fréchet differentiable at each point of D and G is a Lipschitzian operator. Let $\{A_n\}_{n=0}^{\infty}$ be a sequence of bounded linear operators from X into Y such that ${A_n}^{-1} \in B(Y, X)$. To solve the operator equation (1.6), we define the Newton-like algorithm as below: Starting with $x_0 \in D$ and after $x_n \in D$ is defined, we define the next iterate x_{n+1} as follows:

$$x_{n+1} = x_n - {A_n}^{-1}(F(x_n) + G(x_n)), \text{ for all } n \in \mathbb{N}_0. \tag{3.1}$$

Now we are ready to prove the main result for solving Problem (1.6) in the framework of a Banach space.

Theorem 3.1 *Let F and G be two operators defined on an open convex subset D of a Banach space X with values in a Banach space Y such that F is continuously Fréchet differentiable at each point of D and G is a Lipschitzian operator with Lipschitz constant K. Let $\{A_n\}_{n=0}^{\infty}$ be a sequence of bounded linear operators from X into Y. For $x_0 \in D$, assume that the operators F and $\{A_n\}$ satisfy (1.5) with $\omega \in \Phi$ and the following conditions:*

(*C1*) ${A_n}^{-1} \in B(Y, X)$ *and* $\|{A_n}^{-1}\| \leq \beta$ *for all* $n \in \mathbb{N}_0$ *and for some* $\beta > 0$*;*
(*C2*) $\|{A_0}^{-1}(F(x_0) + G(x_0))\| \leq \eta$*, for some* $\eta > 0$;
(*C3*) $\|F'_{x_0} - A_n\| \leq \delta$*, for all* $n \in \mathbb{N}_0$ *and for some* $\delta > 0$*;*
(*C4*) *The function $h(r)$ defined by*

$$h(r) = \int_0^r \beta(\omega(t) + K + \delta)dt - r + \eta \text{ for all } r \in [0, \infty)$$

has a minimum positive zero r^* in $(0, \infty)$.

Assume that $B_{r^*}[x_0] \subseteq D$. Then, we have the followings:

(*a*) *The sequence $\{x_n\}$ generated by (3.1) is well defined, remains in $B_{r^*}[x_0]$ and converges to a solution x^* of (1.6). The following error estimates hold:*

$$\|x_n - x_{n-1}\| \leq t_n - t_{n-1}, \text{for all } n \in \mathbb{N}, \tag{3.2}$$

$$\|x_n - x_0\| \leq t_n, \text{ for all } n \in \mathbb{N} \tag{3.3}$$

and

$$\|x^* - x_n\| \le r^* - t_n, \textit{for all } n \in \mathbb{N}_0,$$

where $\{t_n\}$ *is a sequence generated by*

$$t_0 = 0, t_{n+1} = t_n + h(t_n), n \in \mathbb{N}_0. \tag{3.4}$$

(b) *If there exists a point* $r^{**} \in (0, \infty)$ *such that* $h(r^{**}) \le 0$ *and* r^* *is the unique zero of* $h(r)$ *in* $(0, r^{**}]$*, then* x^* *is the unique solution of* (1.6) *in* $B_{r^{**}}[x_0] \cap D$.

Proof (a) Set $a = \beta, b = K + \delta$ and $c = \eta$. It follows from Lemma 2.2 that the sequence $\{t_n\}$ defined by (3.4) is monotonically increasing and converges to r^*. We now proceed with the following steps:

Step 1. $\|x_n - x_{n-1}\| \le t_n - t_{n-1}$ and $\|x_n - x_0\| \le t_n$ for all $n \in \mathbb{N}$.

Since $\|x_1 - x_0\| = \|{A_0}^{-1}(F(x_0) + G(x_0))\| \le \eta = t_1 - t_0$. Hence (3.2) and (3.3) hold for $n = 1$. Let $k > 1$ be a fixed integer. Assume that (3.2) and (3.3) hold for $n = 1, 2, \cdots, k$. Again, from (3.1), we have

$$\begin{aligned}
\|x_{k+1} - x_k\| &= \Big\|{A_k}^{-1}(F(x_k) + G(x_k) - F(x_{k-1}) - G(x_{k-1}) \\
&\qquad -A_{k-1}(x_k - x_{k-1}))\Big\| \\
&= \Big\|{A_k}^{-1}(F(x_k) - F(x_{k-1}) - F'_{x_0}(x_k - x_{k-1}) + G(x_k) \\
&\qquad -G(x_{k-1}) + F'_{x_0}(x_k - x_{k-1}) - A_{k-1}(x_k - x_{k-1}))\Big\| \\
&\le \Big\|{A_k}^{-1}\Big\| \left[\Big\|F(x_k) - F(x_{k-1}) - F'_{x_0}(x_k - x_{k-1})\Big\|\right. \\
&\qquad + \|G(x_k) - G(x_{k-1})\| + \Big\|F'_{x_0}(x_k - x_{k-1}) \\
&\qquad \left. - A_{k-1}(x_k - x_{k-1})\Big\|\right] \\
&\le \Big\|{A_k}^{-1}\Big\| \left[\int_0^1 \Big\|\left(F'_{x_{k-1}+t(x_k - x_{k-1})} - F'_{x_0}\right)(x_k - x_{k-1})\Big\|\right. \\
&\qquad \times dt + \|G(x_k) - G(x_{k-1})\| \\
&\qquad \left. + \|F'_{x_0} - A_{k-1}\|\|x_k - x_{k-1}\|\right] \\
&\le \beta \int_0^1 \omega\left(t\,\|x_k - x_0\| + (1-t)\,\|x_{k-1} - x_0\|\right) \\
&\qquad \times \|x_k - x_{k-1}\|dt + \beta(K+\delta)\|x_k - x_{k-1}\| \\
&\le \int_0^1 \beta\omega(tt_k + (1-t)t_{k-1})(t_k - t_{k-1})dt \\
&\qquad +\beta(K+\delta)(t_k - t_{k-1}) \\
&= \int_{t_{k-1}}^{t_k} \beta(\omega(t) + K + \delta)dt = t_{k+1} - t_k
\end{aligned}$$

and hence

$$\|x_{k+1} - x_0\| \leq \|x_{k+1} - x_k\| + \|x_k - x_{k-1}\| + \cdots + \|x_1 - x_0\| \leq t_{k+1} - t_0.$$

Thus, (3.2) and (3.3) hold for $n = k + 1$. Hence, (3.2) and (3.3) hold for all $n \in \mathbb{N}$.

Step 2. *$\{x_n\}$ converges to a solution $x^* \in B_{r^*}[x_0]$ of (1.6) with the estimate $\|x^* - x_n\| \leq r^* - t_n$.* Note $\|x_n - x_0\| \leq t_n < r^*$ for all $n \in \mathbb{N}$. This shows that $\{x_n\}$ is in $B_{r^*}[x_0]$. From (3.2), we see that $\{x_n\}$ is a Cauchy sequence and hence converges to some $x^* \in B_{r^*}[x_0]$. From (3.1), we get

$$\begin{aligned} \|F(x_n) + G(x_n)\| &\leq \|F'_{x_0} - A_n\|\|x_{n+1} - x_n\| + \|F'_{x_0}\|\|x_{n+1} - x_n\| \\ &\leq (\delta + \|F'_{x_0}\|)(t_{n+1} - t_n) \to 0 \text{ as } n \to \infty. \end{aligned}$$

From the continuity of F and G, we have $F(x^*) + G(x^*) = 0$. From (3.2), we have

$$\|x_{m+n} - x_n\| \leq \sum_{k=n+1}^{m+n} \|x_k - x_{k-1}\| \leq t_{m+n} - t_n.$$

Taking limit as $m \to \infty$, we get $\|x^* - x_n\| \leq r^* - t_n$.

(b) Suppose there exists a point $r^{**} \in (0, \infty)$ such that $h(r^{**}) \leq 0$ and r^* is the unique zero of $h(r)$ in $(0, r^{**}]$. We have seen in part (a) that $x^* \in B_{r^{**}}[x_0]$ is a solution of (1.6). Suppose that $y^* \in B_{r^{**}}[x_0]$ is another solution of (1.6). Define a sequence $\{s_n\}$ by

$$s_{n+1} = s_n + h(s_n),\ s_0 = r^{**},\ n \in \mathbb{N}_0.$$

By Lemma 2.2, it follows that the sequence $\{s_n\}$ is monotonically decreasing and converges to r^*. We now show that

$$\|y^* - x_n\| \leq s_n - t_n,\ n \in \mathbb{N}_0. \tag{3.5}$$

For $n = 0$, (3.5) holds, since $\|y^* - x_0\| \leq r^{**} = s_0 - t_0$. Let $k > 0$ be a fixed positive integer. Suppose that (3.5) holds for $n = 0, 1, \cdots, k$. From (3.1), we obtain

$$\begin{aligned}
\|y^* - x_{k+1}\| &= \|y^* - x_k + {A_k}^{-1}(F(x_k) + G(x_k))\| \\
&= \|{A_k}^{-1}(F(x_k) + G(x_k) - F(y^*) - G(y^*) - A_k(x_k - y^*))\| \\
&= \|{A_k}^{-1}(F(x_k) - F(y^*) - F'_{x_0}(x_k - y^*) + G(x_k) - G(y^*) \\
&\quad + F'_{x_0}(x_k - y^*) - A_k(x_k - y^*))\| \\
&\leq \|{A_k}^{-1}\| \Bigg[\int_0^1 \|(F'_{y^*+t(x_k-y^*)} - F'_{x_0})(x_k - y^*)\| dt \\
&\qquad\qquad + \|G(x_k) - G(y^*)\| + \|F'_{x_0} - A_k\| \|x_k - y^*\| \Bigg] \\
&\leq \beta \int_0^1 \omega((1-t)\|y^* - x_k\| + \|x_k - x_0\|)\|y^* - x_k\| dt \\
&\quad + \beta(K + \delta)\|y^* - x_k\| \\
&\leq \int_0^1 \beta\omega((1-t)(s_k - t_k) + t_k)(s_k - t_k)dt + \beta(K + \delta)(s_k - t_k) \\
&= \int_{t_k}^{s_k} \beta(\omega(t) + K + \delta)dt \\
&= \int_0^{s_k} \beta(\omega(t) + K + \delta)dt - \int_0^{t_k} \beta(\omega(t) + K + \delta)dt \\
&= s_{k+1} - t_{k+1}.
\end{aligned}$$

Thus, (3.5) holds for $n = k + 1$. Hence, (3.5) holds for all $n \in \mathbb{N}_0$. Then, in view of (3.5), we get

$$\|y^* - x_n\| \leq s_n - t_n \to 0 \text{ as } n \to \infty.$$

Hence, $x^* = y^*$. Thus, x^* is the unique solution of (1.6) in $B_{r^{**}}[x_0]$.

□

Remark 3.2 Bartle [11] pointed out that algorithm (1.2) can be applied for solving (1.1) and Rheinboldt [23] applied the algorithm (1.2) for solving (1.1). As the operator G in (1.6) is not necessarily differentiable, so we can not solve the operator equation (1.6) by applying the results of Bartle [11] and Rheinboldt [23]. Theorem 3.1 is applicable for solving the operator equation (1.6). Therefore, Theorem 3.1 is more general than results of Bartle [11] and Rheinboldt [23].

Remark 3.3 Consider a sequence $\{z_n\}_{n=0}^{\infty}$ in D and define $A_n = A(z_n)$, for all $n \in \mathbb{N}_0$. In this case, the iterative algorithm (3.1) reduces to

$$x_{n+1} = x_n - A(z_n)^{-1}(F(x_n) + G(x_n)), n \in \mathbb{N}_0. \tag{3.6}$$

Suppose all the assumptions of Theorem 3.1 hold, then $\{x_n\}$ defined by (3.6) converges to a solution x^* of (1.6).

Remark 3.4 If $A(z_n) = F'_{z_n}$ and $G = 0$, then (3.6) reduces to

$$x_{n+1} = x_n - F'^{-1}_{z_n} F(x_n), n \in \mathbb{N}_0. \tag{3.7}$$

In particular, if $z_n = x_0$, then (3.7) reduces to

$$x_{n+1} = x_n - F'^{-1}_{x_0} F(x_n), n \in \mathbb{N}_0. \tag{3.8}$$

The convergence analysis of (3.7) has been discussed in [11, 30]. The convergence analysis of (3.8) has been discussed in [2, 11].

We now impose some conditions on A_n for existence of A_n^{-1}.

Theorem 3.5 *Let F and G be two operators defined on an open convex subset D of a Banach space X with values in a Banach space Y such that F is continuously Fréchet differentiable at each point of D and G is a Lipschitzian operator with Lipschitz constant K. Let $\{A_n\}_{n=0}^{\infty}$ be a sequence of bounded linear operator from X into Y such that ${A_0}^{-1} \in B(Y, X)$. For $x_0 \in D$, assume that the operator F and $\{A_n\}_{n=0}^{\infty}$ satisfy (1.5) with $\omega \in \Phi$ and $(C2) - (C3)$ with the following conditions:*

(*C5*) $\|A_0 - A_n\| \leq \beta'$ *and* $\beta' \|{A_0}^{-1}\| < 1$ *for all* $n \in \mathbb{N}_0$ *and for some* $\beta' > 0$*;*
(*C6*) *The function $h_0(r)$ defined by*

$$h_0(r) = \int_0^r \beta''(\omega(t) + K + \delta)dt - r + \eta, r > 0$$

has a minimum positive zero $r^* \in (0, \infty)$, where $\beta'' = \max\left\{\frac{1}{\beta'}, \frac{\|{A_0}^{-1}\|}{1-\beta'\|{A_0}^{-1}\|}\right\}$.

Assume that $B_{r^*}[x_0] \subseteq D$. Then, we have the followings:

(*a*) *The sequence $\{x_n\}$ generated by (3.1) is well defined, remains in $B_{r^*}[x_0]$ and converges to a solution x^* of (1.6). The following error estimates hold:*

$$\|x_{n+1} - x_n\| \leq r_{n+1} - r_n$$

and

$$\|x^* - x_n\| \leq r^* - r_n,$$

where $\{r_n\}$ is a sequence generated by

$$r_0 = 0, r_{n+1} = r_n + h_0(r_n).$$

(*b*) *If there exists a point $r^{**} \in (0, \infty)$ such that $h_0(r^{**}) \leq 0$ and r^* is the unique zero of $h_0(r)$ in $(0, r^{**}]$, then x^* is the unique solution of (1.6) in $B_{r^{**}}[x_0] \cap D$.*

Proof Note

$$\|A_0 - A_n\| \leq \beta' < \frac{1}{\|{A_0}^{-1}\|}.$$

Hence by Lemma 2.1, ${A_n}^{-1}$ exists for each $n \in \mathbb{N}$ and

$$\|{A_n}^{-1}\| \leq \frac{\|{A_0}^{-1}\|}{1 - \beta'\|{A_0}^{-1}\|}.$$

Thus,

$$\|{A_n}^{-1}\| \leq \beta'', \text{ for all } n \in \mathbb{N}_0.$$

Hence all the conditions of Theorem 3.1 hold for $\beta = \beta''$. Therefore, the conclusion of Theorem 3.5 follows from Theorem 3.1. □

Remark 3.6 Consider a sequence $\{z_n\}_{n=0}^{\infty}$ in D and define $A_n = A(z_n)$, for all $n \in \mathbb{N}_0$. Suppose all the assumptions of Theorem 3.5 hold. Then $\{x_n\}$ defined by (3.6) converges to a solution x^* of (1.6).

4 Application

Let X be a Banach space over the field $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$) with norm $\|\cdot\|$ and D an open convex subset of X. Further, let $B(X)$ be the Banach space of bounded linear operators from X into itself. Let $S \in B(X)$, $u \in X$ and $\lambda \in \mathbb{F}$. We seek a solution $x \in X$ of the nonlinear Fredholm-type operator equation:

$$x - \lambda S(Q(x) + R(x)) = u, \tag{4.1}$$

where $Q : D \to X$ is a continuously Fréchet differentiable at each point of D and $R : D \to X$ is a Lipschitzian operator. The operator equation of the form (4.1) with $R = 0$ is discussed in [16, 18]. Define the operators $F, G : D \to X$ by

$$F(x) = x - \lambda SQ(x) - u, x \in D \tag{4.2}$$

and

$$G(x) = -\lambda SR(x), x \in D. \tag{4.3}$$

Then, solving problem (4.1) is equivalent to solving the equation (1.6). From (4.2), we obtain

$$F'_x(y) = y - \lambda SQ'_x(y), y \in X.$$

We now apply Theorem 3.5 for solving operator equation (4.1).

Theorem 4.1 *Let D be an open convex subset of a Banach space X and let Q and R be two operators from X into itself such that Q is continuously Fréchet differentiable at each point of D and R is a Lipschitzian operator with Lipschitz constant K^*. Let $S \in B(X)$, $u \in X$ and $\lambda \in \mathbb{F}$. Let $\{A_n\}_{n=0}^{\infty}$ be a sequence in $B(X)$ such that A_0^{-1} exists. For $x_0 \in D$, assume that the operator Q and $\{A_n\}$ satisfy the conditions $(C5)$ and*

- *(C7)* $\|A_0^{-1}(x_0 - \lambda S(Q(x_0) + R(x_0)) - u)\| \leq \eta$ *for some* $\eta > 0$*;*
- *(C8)* $\|I - \lambda SQ'_{x_0} - A_n\| \leq \delta$ *for some* $\delta > 0$*;*
- *(C9)* $\|Q'_x - Q'_{x_0}\| \leq \omega_0(\|x - x_0\|)$ *for all* $x \in D$*, where* $\omega_0 \in \Phi$*;*

$(C10)$ *The function*

$$h_0(r) = \int_0^r \beta''(|\lambda|\|S\|(\omega_0(t) + K^*) + \delta)dt - r + \eta, \ r > 0$$

has a minimum positive zero $r^* \in (0, \infty)$, *where* $\beta'' = \max\left\{\frac{1}{\beta'}, \frac{\|A_0{}^{-1}\|}{1-\beta'\|A_0{}^{-1}\|}\right\}$.

Assume that $B_{r^*}[x_0] \subseteq D$. *Then, we have the followings:*

(a) *The sequence* $\{x_n\}$ *generated by*

$$x_{n+1} = x_n - A_n^{-1}(x_n - \lambda S(Q(x_n) + R(x_n)) - u), n \in \mathbb{N}_0 \tag{4.4}$$

is well defined, remains in $B_{r^*}[x_0]$ *and converges to a solution* x^* *of* (4.1). *The following error estimates hold:*

$$\|x_{n+1} - x_n\| \le r_{n+1} - r_n \tag{4.5}$$

and

$$\|x^* - x_n\| \le r^* - r_n, \tag{4.6}$$

where $\{r_n\}$ *is a sequence generated by*

$$r_0 = 0, r_{n+1} = r_n + h_0(r_n).$$

(b) *If there exists a point* $r^{**} \in (0, \infty)$ *such that* $h_0(r^{**}) \le 0$ *and* r^* *is the unique zero of* $h_0(r)$ *in* $(0, r^{**}]$, *then* x^* *is the unique solution of* (4.1) *in* $B_{r^{**}}[x_0] \cap D$.

Proof Let $F, G : D \to X$ be two operators defined by (4.2)–(4.3). Clearly, F is Fréchet differentiable at each point of D and G is a Lipschitzian with Lipschitz constant $|\lambda|\|S\|K^*$. From $(C7)$, we have

$$\|A_0^{-1}(F(x_0) + G(x_0))\| = \|A_0^{-1}(x_0 - \lambda S(Q(x_0) + R(x_0)) - u)\| \le \eta,$$

hence $(C2)$ is satisfied. From $(C8)$, we have

$$\|F'_{x_0} - A_n\| = \|I - \lambda S Q'_{x_0} - A_n\| \le \delta,$$

it follows that condition $(C3)$ holds. For $x \in D$, we have that

$$\begin{aligned}\|F'_x - F'_{x_0}\| &= \sup\{\|F'_x z - F'_{x_0} z\| : z \in X, \|z\| = 1\}\\ &\le |\lambda|\|S\| \sup\{\|Q'_x - Q'_{x_0}\|\|z\| : z \in X, \|z\| = 1\}\\ &\le \omega(\|x - x_0\|),\end{aligned}$$

where $\omega(t) = |\lambda|\|S\|\omega_0(t)$. Thus, all the assumptions of Theorem 3.5 are satisfied. Note Algorithm 4.4 can be written as (3.1). Therefore, Theorem 4.1 follows from Theorem 3.5. □

The following corollary follows from Theorem 4.1.

Corollary 4.2 *Let* D *be an open convex subset of a Banach space* X *and let* Q *and* R *be two operators from* X *into itself such that* Q *is continuously Fréchet differentiable at each point of* D *and* R *is a Lipschitzian operator with Lipschitz constant* K^*. *Let* $S \in B(X)$, $u \in X$ *and* $\lambda \in \mathbb{F}$. *Let* $\{z_n\}$ *be a sequence in* D *and* A *be an operator from*

X into $B(X)$ such that $A(z_0)^{-1}$ exists. For $x_0 \in D$, assume that the operator Q and $A(z_n)$ satisfy $(C9)$ with $\omega_0 \in \Phi$ and

(C11) $\|A(z_0) - A(z_n)\| \leq \beta'$ *and* $\beta'\|A(z_0)^{-1}\| < 1$ *for some* $\beta' > 0$*;*
(C12) $\|A(z_0)^{-1}(x_0 - \lambda S(Q(x_0) + R(x_0)) - u)\| \leq \eta$ *for some* $\eta > 0$*;*
(C13) $\|I - \lambda S Q'_{x_0} - A(z_n)\| \leq \delta$ *for some* $\delta > 0$*;*
(C14) *The function*

$$h_0(r) = \int_0^r \beta''(|\lambda|\|S\|(\omega_0(t) + K^*) + \delta)dt - r + \eta, \;\; r > 0$$

has a minimum positive zero $r^* \in (0, \infty)$*, where* $\beta'' = \max\left\{\frac{1}{\beta'}, \frac{\|A(z_0)^{-1}\|}{1-\beta'\|A(z_0)^{-1}\|}\right\}$.

Assume that $B_{r^*}[x_0] \subseteq D$. Then, we have the following:

(a) *The sequence $\{x_n\}$ generated by*

$$x_{n+1} = x_n - A(z_n)^{-1}(x_n - \lambda S(Q(x_n) + R(x_n)) - u), n \in \mathbb{N}_0 \tag{4.7}$$

is well defined, remains in $B_{r^*}[x_0]$ and converges to a solution x^* of (4.1). The error estimates (4.5) and (4.6) hold.

(b) *If there exists a point $r^{**} \in (0, \infty)$ such that $h_0(r^{**}) \leq 0$ and r^* is the unique zero of $h_0(r)$ in $(0, r^{**}]$, x^* is the unique solution of (4.1) in $B_{r^{**}}[x_0] \cap D$.*

We now discus the solution of Fredholm integral equation.

Example 4.3 Let $D = X = C[0, 1]$ be the space of all continuous real valued functions defined on $[0, 1]$ with norm $\|x\| = \sup\limits_{s\in[0,1]} |x(s)|$. Let $\lambda = \frac{1}{20}$ and consider the following nonlinear integral equation

$$x(s) = \sin(\pi s) + \lambda \int_0^1 e^{st}(\sin(x(t)) + |x(t)|)dt, s \in [0, 1]. \tag{4.8}$$

Define $S, Q, R : C[0, 1] \to C[0, 1]$ by

$$Sx(s) = \int_0^1 e^{st}x(t)dt, \; Qx(s) = \sin(x(s)), \; Rx(s) = |x(s)|, \;\text{ for all } x \in C[0, 1] \text{ and } s \in [0, 1].$$

Clearly, S is a bounded linear operator on $C[0, 1]$ with $\|S\| = e - 1$ and Q is a Fréchet differentiable operator on D with its Fréchet derivative Q'_x given by $Q'_x h(s) = \cos(x(s))h(s)$ for all $h \in C[0, 1]$ and $s \in [0, 1]$. Also, R is a Lipschitzian operator with Lipschitz constant $K^* = 1$. Let $x_0(s) = \sin(\pi s)$. Then $x_0 \in C[0, 1]$. For any $x \in C[0, 1]$, we have

$$\begin{aligned}\|(Q'_x - Q'_{x_0})h\| &= \sup_{s\in[0,1]} |(Q'_x - Q'_{x_0})h(s)| \\ &= \sup_{s\in[0,1]} |(\cos(x(s)) - \cos(x_0(s)))h(s)| \\ &\leq \|x - x_0\|\|h\|.\end{aligned}$$

Thus, we have

$$\|Q'_x - Q'_{x_0}\| \leq \omega_0(\|x - x_0\|), \text{ for all } x \in C[0,1],$$

where $\omega_0(t) = t$, $t \geq 0$. Clearly, $\omega_0 \in \Phi$. Let $\{z_n\}$ be a sequence in $C[0,1]$ defined by

$$z_n(s) = \sin\left(\frac{(9n+10)\pi s}{10(n+1)}\right), n \in \mathbb{N}_0.$$

Observe that

$$\|z_n - x_0\| = \sup_{s\in[0,1]}\left|\sin\left(\frac{(9n+10)\pi s}{10(n+1)}\right) - \sin(\pi s)\right| \leq \sup_{s\in[0,1]}\frac{n\pi s}{10(n+1)} = \frac{n\pi}{10(n+1)}.$$

Next, consider the sequence $\{A(z_n)\}$ given by

$$A(z_n)h(s) = h(s) - \lambda\int_0^1 \overline{k}(s,t)\cos(z_n(t))h(t)dt, \text{ for all } h \in C[0,1] \text{ and } s \in [0,1],$$

where $\overline{k}(s,t) = 1 + st + \frac{s^2t^2}{2}$. Now, we have

$$A(z_0)h(s) = h(s) - \lambda\int_0^1 \overline{k}(s,t)\cos(z_0(t))h(t)dt, \text{ for all } s \in [0,1].$$

It follows from [16] that $A(z_0)^{-1}$ exists and $\|A(z_0)^{-1}\| \leq \frac{12}{11}$. Next, we have

$$\begin{aligned}
|(A(z_0) - A(z_n))h(s)| &= \left|\lambda\int_0^1 \overline{k}(s,t)(\cos(z_n(t)) - \cos(z_0(t)))h(t)dt\right| \\
&\leq \lambda\int_0^1 \left|\overline{k}(s,t)(\cos(z_n(t)) - \cos(z_0(t)))\right| |h(t)|dt \\
&\leq \frac{(6+3s+s^2)}{120}\|z_n - z_0\|\|h\|,
\end{aligned}$$

which gives

$$\|(A(z_0) - A(z_n))h\| = \sup_{s\in[0,1]}|(A(z_0) - A(z_n))h(s)| \leq \frac{1}{12}\|z_n - z_0\|\|h\|.$$

Hence, we have

$$\|A(z_0) - A(z_n)\| \leq \frac{1}{12}\|z_n - z_0\| \leq \frac{n\pi}{120(n+1)} \approx 0.0262 < 0.40 = \beta'.$$

Clearly, $\beta' \|A(z_0)^{-1}\| < 1$. Using Gauss-Legendre numerical integration formula with two nodes, we obtain $\|A(z_0)^{-1}(x_0 - \lambda S(Q(x_0) + R(x_0)) - u)\| < 0.12 = \eta$. Now, we have

$$\begin{aligned}
|(I-\lambda SQ'_{x_0}-A(z_n))h(s)| &= \left|\lambda \int_0^1 \left[\, \overline{k}(s,t)\cos(z_n(t)) - e^{st}\cos(x_0(t))\right] h(t)dt\right| \\
&\le \lambda \int_0^1 \left|\overline{k}(s,t)\cos(z_n(t)) - \left(\overline{k}(s,t) + \sum_{k=3}^{\infty} \frac{s^k t^k}{k!}\right)\cos(x_0(t))\right| |h(t)|\, dt \\
&\le \lambda \int_0^1 \Bigg[\left|\overline{k}(s,t)(\cos(z_n(t)) - \cos(x_0(t)))\right| \\
&\qquad + \left|\sum_{k=3}^{\infty} \frac{s^k t^k}{k!}\cos(x_0(t))\right|\Bigg] |h(t)|\, dt \\
&\le \lambda \int_0^1 \left[\left|\overline{k}(s,t)(\cos(z_n(t)) - \cos(x_0(t)))\right| + \frac{s^3 t^3 e^{st}}{6}\right] |h(t)|\, dt \\
&\le \lambda \left[\|z_n - x_0\| \left(1 + \frac{s}{2} + \frac{s^2}{6}\right) + \frac{e}{24}\right] \|h\|,
\end{aligned}$$

which gives that

$$\begin{aligned}
\|(I - \lambda SQ'_{x_0} - A(z_n))h\| &= \sup_{s\in[0,1]} |(I - \lambda SQ'_{x_0} - A(z_n))h(s)| \\
&\le \sup_{s\in[0,1]} \lambda \left[\|z_n - x_0\| \left(1 + \frac{s}{2} + \frac{s^2}{6}\right) + \frac{e}{24}\right] \|h\| \\
&= \left[\frac{1}{12}\|z_n - x_0\| + \frac{e}{480}\right] \|h\|.
\end{aligned}$$

Hence, we have

$$\|I - \lambda SQ'_{x_0} - A(z_n)\| \le \frac{1}{12}\|z_n - x_0\| + \frac{e}{480} \le \frac{n\pi}{120(n+1)} + \frac{e}{480} < 0.032 = \delta.$$

Since $\frac{\|A(z_0)^{-1}\|}{1-\beta'\|A(z_0)^{-1}\|} \le \dfrac{2}{11-\pi} < 2.5 = \frac{1}{\beta'}$, we have $\beta'' = \frac{1}{\beta'} = 2.5$. Note that

$$\begin{aligned}
h_0(r) &= \int_0^r \beta''(\lambda\|S\|(\omega_0(t) + K^*) + \delta)dt - r + \eta \\
&= \beta''\left(\lambda\|S\|\left(\frac{r^2}{2} + r\right) + \delta r\right) - r + \eta \\
&= \frac{\beta''(e-1)}{40}r^2 + \frac{\beta''(e-1) + 20\beta''\delta - 20}{20}r + \eta \\
&= Ar^2 + Br + C,
\end{aligned}$$

where $A = \frac{\beta''(e-1)}{40}$, $B = \frac{\beta''(e-1)+20\beta''\delta-20}{20}$, $C = \eta$. Since $B^2 - 4AC \approx 0.4458 > 0$, the equation $h_0(r) = 0$ has real and distinct roots, which are $\frac{-B\pm\sqrt{B^2-4AC}}{2A}$. For $r^* = \frac{-B-\sqrt{B^2-4AC}}{2A}$ and $r^{**} = 6.38 < \frac{-B+\sqrt{B^2-4AC}}{2A}$, one can see that r^* is the minimum positive zero of $h_0(r)$, $h_0(r^{**}) \leq 0$ and r^* is the unique zero of $h_0(r)$ in $(0, r^{**}]$. Hence, all the conditions of Corollary 4.2 are satisfied. Therefore, the sequence $\{x_n\}$ generated by (4.7) is well defined, remains in $B_{r^*}[x_0]$ and converges to a solution of the integral equation (4.8) which is unique in $B_{r^{**}}[x_0]$.

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