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S-ITERATION PROCESS OF NEWTON-LIKE AND APPLICATIONS

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Abstract. In this paper, we introduce a variant of Newton method for finding the zero of a nonlinear operator in Banach space setting. Our algorithm is based on normal S-iteration process introduced by first author. Our result mainly improves the corresponding results of Argyros [On a theorem of L. V. Kantorovich concerning Newton's method, J. Comput. Appl. Math., 155 (2003), 223-230] and Vijesh and Subrahmanyam [A Newton-type method and its application, Int. J. Math. Math. Sci., (2006), 1-9]. The rate of convergence of our algorithm is faster than the Newton methods studied by Vijesh [1, 2] and Argyros [3].

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1. Introduction

Let X and Y be two Banach spaces. In the sequel, for given $x \in X$ and $r > 0$, $B_r[x]$ (and $B_r(x)$) will denote the set $\{y \in X : \|y-x\| \leq r\}$ (and $\{y \in X : \|y-x\| < r\}$), $B(X,Y)$ will denote the space of all bounded linear operators from X to Y and $\mathbb{N}_0$ will denote the set $\mathbb{N} \cup \{0\}$.

Many mathematical problems such as differential equations, integral equations, economics theory, game theory and variational inequalities can be formulated to fixed point problem (see[4, 5]). This has motivated us to consider the following fixed point problem:

$$\text{to find } x \in D \text{ such that } x = G(x), \tag{1}$$

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where $G : D \subset X \to X$ is nonlinear operator defined on nonempty subset D of a Banach space X. The easiest iterative method for constructing a sequence is Picard iterative method [6] which is given by

$$x_{n+1} = G(x_n), n \in \mathbb{N}_0. \tag{2}$$

Banach contraction principle on fixed points provides sufficient conditions for the convergence of the iterative method (2) to the fixed point of G (see [4, 5, 7]). More detail for approximating of fixed point can be found in [8, 9, 10].

Recently, Agarwal, O'Regan and Sahu [11] have introduced the S-iteration process as follows: Let X be a normed space, C a nonempty convex subset of X and $T : C \to C$ an operator. Then, for an arbitrary $x_0 \in C$, the S-iteration process is defined by

$$\begin{cases} x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n Ty_n, \\ y_n = (1-\beta_n)x_n + \beta_n Tx_n, n \in \mathbb{N}_0, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ satisfying certain conditions. The details of S-iteration process can be found in [4].

Motivated by S-iteration process, the first author (see [12]) has introduced the normal S-iteration process as follows: Let X be a normed space, C a nonempty convex subset of X and $T : C \to C$ an operator. Then, for arbitrary $x_0 \in C$, the normal S-iteration process is defined by

$$x_{n+1} = T[(1-\alpha_n)x_n + \alpha_n Tx_n], n \in \mathbb{N}_0, \tag{3}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$.

Following [12, Theorem 3.6], we have following comparison theorem.

Theorem 1.1. *Let C be a nonempty closed convex subset of a Banach space X and let $T : C \to C$ be a contraction mapping with the Lipschitz constant k. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ such that $\alpha \leq \alpha_n \leq 1$ for all $n \in \mathbb{N}$ and for some $\alpha > 0$. For given $u_1 = v_1 = w_1 \in C$, define the sequences $\{u_n\}, \{v_n\}$ and $\{w_n\}$ in C as follows:*

S-iteration process: $u_{n+1} = T[(1-\alpha_n)u_n + \alpha_n Tu_n]$.
Picard iteration: $v_{n+1} = Tv_n$.
Mann iteration process: $w_{n+1} = (1-\alpha_n)w_n + \alpha_n Tw_n$.

Let x^* be a fixed point of T. Then, we have the following:

(a) $\|u_{n+1} - x^*\| \leq k^n[1-(1-k)\alpha]^n \|u_1 - x^*\|$, *for all* $n \in \mathbb{N}$.
(b) $\|v_{n+1} - x^*\| \leq k^n \|v_1 - x^*\|$, *for all* $n \in \mathbb{N}$.
(c) $\|w_{n+1} - x^*\| \leq [1-(1-k)\alpha]^n \|w_1 - x^*\|$, *for all* $n \in \mathbb{N}$.

Clearly, the normal S-iteration process is faster than the Picard and Mann iteration processes for contraction mappings.

On the other hand, a lot of challenging problems in physics, numerical analysis, engineering and applied mathematics can be formulated in the following form

$$\text{to find } x \in D \text{ such that } F(x) = 0, \tag{4}$$

where D is a nonempty open convex subset of a Banach space X and $F : D \subseteq X \to Y$ is a nonlinear operator. Undoubtedly, Newton method is the most popular method for solving such problems. Starting with $w_0 \in D$, method is given by

$$w_{n+1} = w_n - {F'_{w_n}}^{-1} F(w_n), n \in \mathbb{N}_0, \tag{5}$$

where F'_w denotes the Fréchet derivative of F for $w \in D$. A number of mathematician [13, 14, 15, 16] have generalized and established the local and the semilocal convergence theorems for Newton method (5). Details can be found in Argyros [17], Wu and Zhao [18] and references therein.

In [3], Argyros proved the following theorem for semilocal convergence analysis of (5) to solve the operator equation (4).

Theorem 1.2. *Let F be a continuous operator defined on an open convex subset D of a Banach space X with values in a Banach space Y and continuously Fréchet differentiable at some $w_0 \in D$. Assume that the operator F satisfies the following conditions:*

(i) $F'^{-1}_{w_0} \in B(Y, X)$;
(ii) $\|F'^{-1}_{w_0} F(w_0)\| \leq \eta$ *for some* $\eta > 0$;
(iii) *For each* $\kappa > 0$, *there exists* $r = r(\kappa) > 0$ *such that* $\|F'^{-1}_{w_0}(F'_w - F'_{w_0})\| \leq \kappa$ *for all* $w \in B_r(w_0)$.

Assume that $\frac{1-3\kappa+4\kappa^2}{(1-\kappa)(1-3\kappa)}\eta < r$, *for* $\kappa \in (0, \frac{1}{3})$ *and* $B_r[w_0]$ *is in* D. *Then, the sequence* $\{w_n\}$ *generated by (5) is well defined in* $B_r(w_0)$ *and converges strongly to the unique solution* $w^* \in B_r[w_0]$ *of (4). Moreover, the following error estimate holds:*

$$\|w_{n+1} - w_n\| \leq d_0^n \|w_1 - w_0\|$$

and

$$\|w_n - w^*\| \leq \frac{d_0^n}{1 - d_0} \|w_1 - w_0\|, \tag{6}$$

where $d_0 = \frac{2\kappa}{1-\kappa}$.

Following the ideas of Argyros [3], Vijesh and Subrahmanyam [2] studied the following result under the assumption that F is Gâteaux differentiable.

Theorem 1.3. *Let F be a continuous operator defined on an open convex subset D of a Banach space X with values in a Banach space Y such that F is Gâteaux differentiable at each point in some neighbourhood of $w_0 \in D$. Assume that the operator F satisfies the following conditions:*

(i) $F'^{-1}_{w_0} \in B(Y, X)$;
(ii) $\|F'^{-1}_{w_0} F(w_0)\| \leq \eta$ *for some* $\eta > 0$;
(iii) *For each* $\kappa > 0$, *there exists* $r = r(\kappa) > 0$ *such that* $\|F'^{-1}_{w_0}(F'_w - F'_{w_0})\| \leq \kappa$ *for all* $w \in B_r(w_0)$;
(iv) F'_w *is piecewise hemicontinuous at each* $w \in B_r[w_0]$.

Assume that $\left(\frac{1-2\kappa}{1-3\kappa}\right)\eta < r$ *for* $\kappa \in (0, \frac{1}{3})$ *and* $B_r[w_0]$ *is in* D. *Then, the sequence* $\{w_n\}$ *generated by (5) is well defined in* $B_r(w_0)$ *and converges to unique solution* $w^* \in B_r[w_0]$ *of (4). Moreover, the following error estimates hold:*

$$\|w_{n+1} - w_n\| \leq d_0^{n-1} c_0 \|w_1 - w_0\|$$

and

$$\|w_n - w^*\| \leq \frac{d_0^{n-1} c_0}{1 - d_0} \|w_1 - w_0\|, \tag{7}$$

where $c_0 = \frac{\kappa}{1-\kappa}$ *and* $d_0 = \frac{2\kappa}{1-\kappa}$.

Note that Theorem 1.3 is an improvement upon Theorem 1.2 under weaker assumptions but the rate of convergence Newton-like method (5) in Theorem 1.2 and Theorem 1.3 are same. Therefore, it is natural to raise the following problem.

Problem 1.4. *Is it possible to develop an algorithm for solving (4) whose rate of convergence is faster than (5)?*

Motivated by normal S-iteration process, a new iteration process of Newton-like method for finding the solution of real equation $f(x) = 0$ has been introduced in [19] as follows:

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = (1 - \alpha_n)x_n + \alpha_n y_n, \\ x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}, \end{cases} \tag{8}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$.

In this paper, motivated by normal S-iteration process (3) and (8), we introduce an iterative algorithm which we call S-iteration process of Newton-like. We establish the convergence theory of the proposed algorithm for computing the solution of operator equation (4) in the framework of Banach space. It is shown that the rate of convergence of our algorithm is faster than Newton method (5) which provides an affirmative answer of the Problem 1.4. Our strong convergence theorem extend and improve corresponding results of Argyros [3] and Vijesh and Subrahmanyam [1, 2] in the context of S-iteration process of Newton-like with larger convergence ball and tighter error bounds.

2. Preliminaries

In this section, we give some basic definitions and lemmas which are very usefull in the remaining part of the paper.

Definition 2.1. [1] Let X and Y be normed spaces and C be a nonempty open subset of X. An operator $T : C \to Y$ is said to be hemicontinuous at $x \in C$, if for any $h \in X$ and $\epsilon > 0$ there exists δ such that $\|T(x + th) - T(x)\| < \epsilon$, whenever $|t| \leq \delta$ and $x + th \in C$.

Definition 2.2. [2] Let X and Y be normed spaces and C a nonempty open convex set of X. An operator $T : C \to Y$ is said to be piecewise-hemicontinuous at each $x \in C$, if the function $g : [0, 1] \to Y$ defined by

$$g(t) = T(x + t(y - x)),$$

for any fixed $y \in C$, is piecewise-continuous in $[0, 1]$.

Definition 2.3. Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers that converges to a and b respectively, and assume that there exists

$$l = \lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|}.$$

(a) If $l = 0$, then it can be said that $\{a_n\}$ converges to a faster than $\{b_n\}$ converges to b.
(b) If $0 < l < \infty$, then it can be said that $\{a_n\}$ and $\{b_n\}$ have the same rate of convergence.

Definition 2.4. (Sahu [11]) Let C be a nonempty convex subset of a normed space X and $T : C \to C$ an operator. The operator $G : C \to C$ is said to be S-operator generated by $\alpha \in (0, 1)$ and T if

$$G = T[(1 - \alpha)I + \alpha T], \tag{9}$$

where I is the identity operator.

Lemma 2.5. *(Rall [20], Page 50) Let L be a bounded linear operator on a Banach space X, L^{-1} exists if and only if there is a bounded linear operator M in X such that M^{-1} exists, and*

$$\|M - L\| \quad < \quad \frac{1}{\|M^{-1}\|}.$$

If L^{-1} exists, then

$$\left\|L^{-1}\right\| \leq \frac{\left\|M^{-1}\right\|}{1 - \|1 - M^{-1}L\|} \leq \frac{\left\|M^{-1}\right\|}{1 - \|M^{-1}\| \, \|M - L\|}.$$

In view of [1] and [21], one can establish the following:

Lemma 2.6. *Let F be a Gâteaux differentiable operator at each point of an open convex subset D of a normed linear space X with values in a Banach space Y such that F'_x is piecewise-hemicontinuous at each $x \in B_r(x_0)$ for some $r > 0$ and for some $x_0 \in D$. Then,*

$$F(x) - F(y) = \int_0^1 F'_{y+t(x-y)}(x - y)dt \text{ for all } x, y \in B_r(x_0).$$

Definition 2.7. Let F be an operator defined on an open convex subset D of a Banach space X with values in a Banach space Y. For $x_0 \in D$, assume that F is Gâteaux differentiable at each point in $B_r(x_0)$ is in D for some $r > 0$ such that $F_x'^{-1}$ exists for all $x \in B_r(x_0)$. Define an operator $T : B_r(x_0) \to X$ by

$$Tx = x - F_x'^{-1}Fx \text{ for all } x \in B_r(x_0). \tag{10}$$

The operator T is called Newton operator.

We now give some properties of Newton operator T defined by (10).

Proposition 2.1. *Let F be a continuous operator defined on an open convex subset D of a Banach space X with values in a Banach space Y such that F is Gâteaux differentiable at each point in some neighbourhood of $x_0 \in D$. Assume that the operator F satisfies the following conditions:*

- $(C1)$ $F_{x_0}'^{-1} \in B(Y, X)$;
- $(C2)$ $\|F_{x_0}'^{-1}F(x_0)\| < \eta$ *for some* $\eta > 0$;
- $(C3)$ *For each* $\kappa > 0$*, there exists* $r = r(\kappa) > 0$ *such that* $\|F_{x_0}'^{-1}(F'_x - F'_{x_0})\| \leq \kappa$ *for all* $x \in B_r(x_0)$;
- $(C4)$ F'_x *is piecewise hemicontinuous at each* $x \in B_r(x_0)$.

Assume that $\frac{\eta}{1-3\kappa} < r$ for $\kappa \in (0, \frac{1}{3})$ and $B_r[x_0]$ is in D. Then, we have the following:

(a) $F_x'^{-1}$ *exists for all* $x \in B_r(x_0)$ *and* $\|(F'_{x_0}{}^{-1}F'_x)^{-1}\| \leq \frac{1}{1-\kappa}$.

(b) *The Newton operator T defined by (10) is a self-operator on $B_r(x_0)$.*

Proof: (a) For $x \in B_r(x_0)$, we have

$$\|I - {F'_{x_0}}^{-1} F'_x\| \leq \|{F'_{x_0}}^{-1}(F'_{x_0} - F'_x)\| \leq \kappa < 1.$$

By Lemma 2.5, ${F'_x}^{-1}$ exists and

$$\|({F'_{x_0}}^{-1} F'_x)^{-1}\| \leq \frac{1}{1-\kappa}.$$

(b) Let $x \in B_r(x_0)$. Using Lemma 2.5, we have

$$\begin{aligned}
&\|T(x) - x_0\| \\
&= \|{F'_x}^{-1}[F(x) - F(x_0) - F'_x(x - x_0) + F(x_0)]\| \\
&= \|({F'_{x_0}}^{-1} F'_x)^{-1}[\textstyle\int_0^1 {F'_{x_0}}^{-1}(F'_{x_0+t(x-x_0)} - F'_{x_0} + F'_{x_0} - F'_x) \\
&(x - x_0)dt - {F'_{x_0}}^{-1} F(x_0)\| \\
&\leq \|({F'_{x_0}}^{-1} F'_x)^{-1}\| \textstyle\int_0^1 \|{F'_{x_0}}^{-1}(F'_{x_0+t(x-x_0)} - F'_{x_0})\| + \|{F'_{x_0}}^{-1}(F'_{x_0} - F'_x)\|\|x - \\
&x_0\|dt + \|({F'_{x_0}}^{-1} F'_x)^{-1}\|\|{F'_{x_0}}^{-1} F(x_0)\| \\
&\leq \tfrac{2\kappa}{(1-\kappa)}\|x - x_0\| + \tfrac{\eta}{(1-\kappa)} \\
&< \tfrac{2\kappa r}{(1-\kappa)} + \tfrac{\eta}{(1-\kappa)} < r.
\end{aligned}$$

Therefore, T is a self-operator on $B_r(x_0)$.

3. Main result

In this section, we extend S-iteration process of Newton-like (8) in the framework of Banach spaces and study the semilocal convergence analysis of our iteration process.

Algorithm 3.1. Let F be a continuous operator defined on an open convex subset D of a Banach space X with values into another Banach space Y. For $x_0 \in D$ and for some $r > 0$, suppose that $B_r(x_0)$ is in D and F be a Gâteaux differentiable on $B_r(x_0)$ such that ${F'_x}^{-1}$ exists and $x - {F'_x}^{-1} Fx \in B_r(x_0)$ for each $x \in B_r(x_0)$. For $\alpha \in (0, 1)$, the S-iteration process of Newton-like $\{x_n\}$ on $B_r(x_0)$ is generated by

$$\begin{cases} x_{n+1} = z_n - {F'_{z_n}}^{-1} F(z_n), \\ z_n = (1-\alpha)x_n + \alpha y_n, \\ y_n = x_n - {F'_{x_n}}^{-1} F(x_n), n \in \mathbb{N}_0. \end{cases} \tag{11}$$

Now, we are ready to discuss the semilocal convergence analysis of Algorithm 3.1 for computing solution of operator equation (4) in the framework of Banach space.

Theorem 3.2. *Let F be a continuous operator defined on an open convex subset D of a Banach space X with values in a Banach space Y such that F is Gâteaux differentiable at each point in some neighbourhood of $x_0 \in D$. Assume that the operator F satisfies the conditions $(C1) \sim (C4)$ of Proposition 2.1. Assume further that $d_0 = \frac{2\kappa}{1-\kappa}$ and $\frac{\eta}{1-3\kappa} < r$ for $\kappa \in (0, \frac{1}{3})$ and $B_r[x_0]$ is in D. Then, we have the following:*

(a) *For $\alpha \in (0,1)$, the sequence $\{x_n\}$ generated by (11) is well defined in $B_r(x_0)$ with*

$$\|z_n - x_n\| \leq d_0 \alpha d^{n-1} \|x_1 - z_0\| \tag{12}$$

and

$$\|x_{n+1} - z_n\| \leq d^n \|x_1 - z_0\| \text{ for all } n \in \mathbb{N}, \tag{13}$$

where $d = d_0(1 - \alpha(1 - d_0))$.

(b) *The sequence $\{x_n\}$ generated by (11) converges strongly to the unique solution $x^* \in B_r[x_0]$ of (4).*

(c) *The following error estimate holds:*

$$\|x_n - x^*\| \leq \frac{(d + \alpha d_0) d^{n-1}}{1 - d} \|x_1 - z_0\| \text{ for all } n \in \mathbb{N}. \tag{14}$$

Proof: (a) Using Proposition 2.1 and (11), one can easily observe that the sequence $\{x_n\}$ generated by (11) is well defined in $B_r(x_0)$.
From $(C2)$ and (11), we have

$$\|z_0 - x_0\| = \| - \alpha {F'_{x_0}}^{-1} F(x_0)\| \leq \alpha \eta.$$

Using Lemma 2.6, (11) and $(C3)$, we obtain

$$\begin{aligned}
&\|x_1 - z_0\| \\
&= \|{F'_{z_0}}^{-1}(F(z_0))\| \\
&= \|{F'_{z_0}}^{-1}(F(z_0) - F(x_0) - F'_{x_0}(y_0 - x_0))\| \\
&\leq \|({F'_{x_0}}^{-1} F'_{z_0})^{-1}\| (\textstyle\int_0^1 \|{F'_{x_0}}^{-1}((F'_{x_0+t(z_0-x_0)} - F'_{x_0})(z_0 - x_0) - \\
&(1-\alpha) F'_{x_0}(y_0 - x_0))\| dt) \\
&\leq \|({F'_{x_0}}^{-1} F'_{z_0})^{-1}\| (\textstyle\int_0^1 \|{F'_{x_0}}^{-1}(F'_{x_0+t(z_0-x_0)} - F'_{x_0})(z_0 - x_0)\| dt + \\
&(1-\alpha)\|{F'_{x_0}}^{-1} F(x_0)\|) \\
&\leq \tfrac{\kappa}{1-\kappa}\|z_0 - x_0\| + \tfrac{1-\alpha}{1-\kappa}\eta \\
&\leq \tfrac{(1-\alpha(1-\kappa))\eta}{1-\kappa}.
\end{aligned}$$

Now for $n \in \mathbb{N}$, by using Lemma 2.6, (11) and $(C3)$, we have

$$\|z_n - x_n\|$$

$= \|\alpha(y_n - x_n)\| = \alpha \|{F'_{x_n}}^{-1} F(x_n)\|$
$= \alpha \|{F'_{x_n}}^{-1}(F(x_n) - F(z_{n-1}) - F'_{z_{n-1}}(x_n - z_{n-1})\|$
$\leq \alpha \|({F'_{x_0}}^{-1} F'_{x_n})^{-1}\| \int_0^1 \|{F'_{x_0}}^{-1}(F'_{z_{n-1}+t(x_n-z_{n-1})} - F'_{z_{n-1}})\| \|x_n - z_{n-1}\| dt$
$\leq \frac{2\kappa\alpha}{1-\kappa} \|x_n - z_{n-1}\|$

and

$\|x_{n+1} - z_n\|$
$= \|{F'_{z_n}}^{-1} F(z_n)\|$
$= \|{F'_{z_n}}^{-1}[F(z_n) - F(x_n) - F'_{x_n}(y_n - x_n)]\|$
$= \|{F'_{z_n}}^{-1}[F(z_n) - F(x_n) - F'_{x_n}(z_n - x_n) - F'_{x_n}(y_n - z_n)]\|$
$\leq \|({F'_{x_0}}^{-1} F'_{z_n})^{-1}\| \Big(\int_0^1 \|{F'_{x_0}}^{-1}(F'_{x_n+t(z_n-x_n)} - F'_{x_n})(z_n - x_n)\| dt$
$+ (1-\alpha) \int_0^1 \|{F'_{x_0}}^{-1}(F'_{z_{n-1}+t(x_n-z_{n-1})} - F'_{z_{n-1}})(x_n - z_{n-1})\| dt \Big)$
$\leq \frac{2\kappa}{1-\kappa} \left(\|z_n - x_n\| + (1-\alpha)\|x_n - z_{n-1}\| \right).$

From (**??**) and (3), we have

$$\|z_1 - x_1\| \leq \frac{2\kappa\alpha}{1-\kappa} \|x_1 - z_0\| = d_0 \alpha d^0 \|x_1 - z_0\|$$

and
$\|x_2 - z_1\|$
$\leq d_0(\|z_1 - x_1\| + (1-\alpha)\|x_1 - z_0\|$
$\leq d_0(1 - \alpha(1 - d_0))\|x_1 - z_0\| = d\|x_1 - z_0\|.$
Hence, (12) and (13) hold for $n = 1$. Let $k > 1$ be a fixed integer. Assume that (12) and (13) hold for $n = 1, 2, \cdots, k$.
(b) Using (**??**), one can easily observe that

$$\|z_{k+1} - x_{k+1}\| \leq d_0 \alpha \|x_{k+1} - z_k\| = d_0 \alpha d^k \|x_1 - z_0\|$$

and by (3), we have

$$\begin{aligned} \|x_{k+2} - z_{k+1}\| &\leq d_0(\|z_{k+1} - x_{k+1}\| + (1-\alpha)\|x_{k+1} - z_k\|) \\ &\leq d_0(d_0\alpha + (1-\alpha))\|x_{k+1} - z_k\| \\ &= d\|x_{k+1} - z_k\| \\ &\leq d^{k+1}\|x_1 - z_0\|. \end{aligned}$$

Hence, we conclude that (12) and (13) hold for $k = n + 1$. Thus, (12) and (13) hold for all $n \in \mathbb{N}$.

For $n \geq 1$ and using (12) and (13), we have

$$\|x_{n+1} - x_n\| \leq \|x_{n+1} - z_n\| + \|z_n - x_n\|$$

$$\begin{aligned} &\leq d\|x_n - z_{n-1}\| + d_0\alpha\|x_n - z_{n-1}\| \\ &= (d + d_0\alpha)\|x_n - z_{n-1}\| \\ &\leq (d + d_0\alpha)d^{n-1}\|x_1 - z_0\|. \end{aligned}$$

Let $m \geq 1, n \in \mathbb{N}$. Then, we have

$$\begin{aligned} \|x_{m+n} - x_m\| &\leq \sum_{j=m}^{m+n-1} \|x_{j+1} - x_j\| \\ &\leq \sum_{j=m}^{m+n-1} (d + d_0\alpha)d^{j-1}\|x_1 - z_0\| \\ &= (d + d_0\alpha)\|x_1 - z_0\| \sum_{j=m}^{m+n-1} d^{j-1} \\ &= (d + d_0\alpha)\|x_1 - z_0\| d^{m-1} \sum_{j=0}^{n-1} d^j \\ &= (d + d_0\alpha)\|x_1 - z_0\| d^{m-1} \frac{1 - d^n}{1 - d} \qquad (15) \end{aligned}$$

Note that $0 < d < 1$. It follows that $\{x_n\}$ is a Cauchy sequence in $B_r[x_0]$ and hence it converges to an element $x^* \in B_r[x_0]$.

Now, we show that the (4) has a unique solution in $B_r[x_0]$. Let $c = (\kappa + 1)\|F'_{x_0}\|$. For all $x \in B_r(x_0)$, we have $\|F'_x\| \leq c$. Observe that

$$\begin{aligned} \|F(x_n)\| &= \|F'_{x_n}(x_{n+1} - x_n)\| \\ &\leq \|F'_{x_n}\|\|x_{n+1} - x_n\| \\ &\leq c\|x_{n+1} - x_n\| \to 0 \text{ as } n \to \infty. \end{aligned}$$

Using continuity of F, we have $F(x^*) = 0$. Thus, x^* is a solution of (4).

Next, suppose that x^* and y^* are two solution of (4) in $B_r[x_0]$. Then, by Lemma 2.6, we have

$$0 = F(x^*) - F(y^*) = \int_0^1 F'_{y^* + t(x^* - y^*)}(x^* - y^*)dt.$$

Define an operator $L : X \to Y$ by

$$L(h) = \int_0^1 F'_{y^* + t(x^* - y^*)} h dt \text{ for all } h \in X.$$

Now, we have

$\|{F'_{x_0}}^{-1}(L - F'_{x_0})\|$

$$= \sup\left\{\left\|{F'_{x_0}}^{-1}\left(\int_0^1 F'_{y^*+t(x^*-y^*)}hdt - F'_{x_0}h\right)\right\| : \|h\| \leq 1\right\}$$
$$\leq \sup\left\{\int_0^1 \left\|{F'_{x_0}}^{-1}(F'_{y^*+t(x^*-y^*)} - F'_{x_0})\right\| \|h\|dt : \|h\| \leq 1\right\}$$
$$\leq \kappa < 1.$$

By Lemma 2.5, L is invertible. Since $L(x^* - y^*) = 0$. Hence, $x^* = y^*$.
(c) Taking limit as $n \to \infty$ in (15), we have

$$\|x_m - x^*\| \leq \frac{(d + \alpha d_0)d^{m-1}}{1-d}\|x_1 - z_0\| \text{ for all } m \in \mathbb{N}.$$ □

Remark 1. One can observe from (7) and (14) that

$$d = d_0(1 - \alpha(1 - d_0)) < d_0. \tag{16}$$

The strict inequality (16) shows that the error estimate in Theorem 3.2 is sharper than that of Theorem 1.3. Therefore, Theorem 3.2 is a significant improvement of Theorem 1.3.

Remark 2. In Theorem 3.2 the radius of convergence ball $B_r[x_0]$ is larger than the radius of convergence ball in Theorem 1.3 and the error estimate is tighter than the error estimate in Theorem 1.3.

4. Application to initial value problem

Let $C[0,1]$ be the space of real-valued continuous functions defined on the interval $[0,1]$ with norm $\|x\|_\infty = \sup_{t\in[0,1]} |x(t)|$ and $C^1[0,1]$ be the space of all continuously differentiable real valued functions defined on $[0,1]$ with the norm $\|x\|_1 = \|x\|_\infty + \|x'\|_\infty$.

Consider the following initial value problem:

$$\frac{dy(s)}{ds} = f(s, y(s)), y(0) = 0, 0 \leq s \leq 1. \tag{17}$$

Assume that $f'_2(s, y(s)) = \frac{\partial}{\partial y} f(s, y(s))$, the partial derivative with respect to second component exists for all $(s, y(s)) \in S \subseteq [0,1] \times \mathbb{R}$ and for any $\varepsilon > 0$ there exists $r > 0$ such that

$$\left\|f'_2(s, y(s)) - f'_2(s, y_0(s))\right\| < \varepsilon \text{ for all } y \in B_r[y_0], \tag{18}$$

where $y_0 \in C^1[0,1]$.

Consider the operator $F : C^1[0,1] \to C[0,1]$ defined by

$$F(y)(s) = \frac{dy(s)}{ds} - f(s, y(s)) \text{ for all } y \in C^1[0,1] \text{ for all } s \in [0,1]. \tag{19}$$

Then, solving problem (17) is equivalent to solving the equation (4).

We now establish the following theorem.

Theorem 4.1. *Let $F : C^1[0,1] \to C[0,1]$ be an operator defined by (19). For some $y_0 \in C^1[0,1]$, assume $F'^{-1}_{y_0}$ exists and $f'_2(s, y(s))$ satisfies (18). Let $\theta_1 = \sup_{s\in[0,1]} \left|f'_2(s, y_0(s))\right| < 1$. Assume that*

$$\frac{\varepsilon}{1-\theta_1} \in \left(0, \frac{1}{3}\right) \ \ \textit{and} \ \ \frac{\|F(y_0)\|}{(1-\theta_1-3\varepsilon)} < r.$$

Then, we have the following:

(a) *The initial value problem (17) has a unique solution y^* in $B_r[y_0]$.*
(b) *For $x_0 = y_0$, the sequence $\{x_n\}$ generated by Algorithm 3.1 is in $B_r[y_0]$ and it converges strongly to y^*.*

Proof: First note that the operator F is Fréchet differentiable and hence it is Gâteaux differentiable and its Gâteaux derivative is given by

$$F'_y h(s) = \frac{dh(s)}{ds} - f'_2(s, y(s))h(s) \text{ for all } h \in C^1[0,1].$$

Our goal is to find an upper bound for $F'^{-1}_{y_0}$. Set

$$F'_{y_0} h(s) = \frac{dh(s)}{ds} - f'_2(s, y_0(s))h(s) = u(s).$$

Immediately, we can write $h(s) = F'^{-1}_{y_0} u(s)$ and arrive at the first order linear initial value problem given by

$$\begin{cases} \frac{dh(s)}{ds} = u(s) + f'_2(s, y_0(s))h(s); \\ h(0) = 0. \end{cases} \tag{20}$$

It should be noted that problem (20) is equivalent to the integral equation of Voltera type of second kind (see [22]) given by

$$h(s) = \int_0^s \left[u(\tau) + f'_2(\tau, y_0(\tau))h(\tau)\right] d\tau.$$

Consider the operator $L : C[0,1] \to C[0,1]$ defined by

$$Lh(t) = \int_0^s f'_2(\tau, y_0(\tau))h(\tau)d\tau, h \in C[0,1].$$

Clearly, the operator L is a linear operator. We can write

$$(I - L)(h)(s) = \int_0^s u(\tau)d\tau.$$

Note that the operator L is bounded and satisfies

$$\|L\| \le \theta_1, \text{ where } \theta_1 = \sup_{s\in[0,1]} \left|f'_2(s, y_0(s))\right|.$$

Consequently, by Lemma 2.5, the operator $(I-L)^{-1}$ exists and $\|(I-L)^{-1}\| \leq \frac{1}{1-\theta_1}$. Therefore, $\|F'^{-1}_{y_0}\| \leq \frac{1}{1-\theta_1}$. Next, by (18), we have

$$\|F'^{-1}_{y_0}(F'_y - F'_{y_0})\| \leq \frac{\varepsilon}{1-\theta_1} \text{ for all } y \in B_r[y_0].$$

For $\kappa > 0$, let $\varepsilon = (1-\theta_1)\kappa$. Clearly, $\eta = \frac{\|F(y_0)\|}{1-\theta_1}$ and $\kappa = \frac{\varepsilon}{1-\theta_1}$ and for $\alpha \in (0,1)$, $\frac{\eta}{1-3\kappa} < r$. Thus, all the assumptions of Theorem 3.2 are satisfied. Therefore, Theorem 4.1 follows from Theorem 3.2. □

5. Numerical Examples

Example 5.1. Let $X = Y = \mathbb{R}, D = (-1,1)$ and define a function F on X by

$$F(x) = x^2, x \in X.$$

For $x_0 = w_0 = \frac{1}{2}$ and $\alpha = \frac{1}{2}$, the sequence $\{w_n\}$ defined by (5) and the sequence $\{x_n\}$ defined by (11) reduces to $w_n = \frac{1}{2^{n+1}}$ and $x_n = \frac{1}{2}\left(\frac{3}{2^3}\right)^n$ for all $n \geq 0$, respectively. It is obvious that $w_n \to 0, x_n \to 0$ as $n \to \infty$. Observe that

$$l = \lim_{n\to\infty} \frac{|x_n|}{|w_n|} = \lim_{n\to\infty} \left(\frac{3}{4}\right)^{n+1} = 0.$$

Therefore, by Definition 2.3, the rate of convergence of the sequence $\{x_n\}$ generated by S-iteration process of Newton-like (11) is sharper than that of the sequence $\{w_n\}$ generated by Newton method (5). In figure 1, we see graphically that the $\{x_n\}$ is faster than $\{w_n\}$.

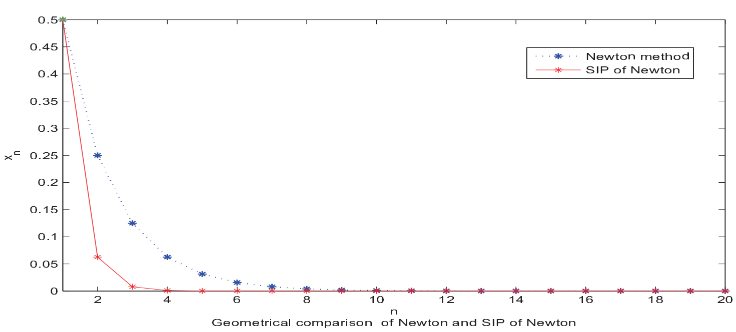

FIGURE 1. Comparison of iteration processes defined by (5) and (11).

Example 5.2. Let $X = Y = \mathbb{R}, D = (0, 1.9)$ and $F : D \to \mathbb{R}$ a function defined by

$$F(x) = \begin{cases} x^2 cos(\frac{1}{x}) - 1.6, & x \neq 0, \\ -1.6, & x = 0. \end{cases}$$

Then, F is Gâteaux differentiable function on D and its derivative at point $x \in D$ is

$$F'_x = \begin{cases} 2x cos(\frac{1}{x}) + sin(\frac{1}{x}), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Let $x_0 = 1.5 \in D$, $\alpha = 0.5$ $r = 0.3$. Then, we have

$$\|{F'_{x_0}}^{-1} F(x_0)\| \leq 0.056533787333943 = \eta,$$

$$\|{F'_{x_0}}^{-1}(F'_x - F'_{x_0})\| < 0.25 = \kappa$$

and

$$d_0 = \frac{2\kappa}{1-\kappa} = 0.67 \; and \; \frac{\eta}{1-3\kappa} = 0.226135149335772 < 0.3$$

Thus, all the assumptions of Theorem 3.2 are satisfied For this function sequence generated by (11) is

$$\begin{cases} x_{n+1} = z_n - \frac{1}{(2z_n cos(\frac{1}{z_n}) + sin(\frac{1}{z_n}))}(z_n^2 cos(\frac{1}{z_n}) - 1.6), \\ z_n = (1-\alpha)x_n + \alpha y_n \\ y_n = x_n - \frac{1}{(2x_n cos(\frac{1}{x_n}) + sin(\frac{1}{x_n}))}(x_n^2 cos(\frac{1}{x_n}) - 1.6). \end{cases}$$

Following table shows that SIP of Newton-like $\{x_n\}$ is faster than Newton $\{w_n\}$.

n	Newton Method	SIP Newton-like y	SIP Newton-like z	SIP Newton-like x
1	1.500000000000000	1.500000000000000	1.471733106333029	1.442624829364997
2	1.443466212666057	1.442320707710629	1.442472768537813	1.442320682775942
3	1.442321145807560	1.442320674460664	1.442320678618303	1.442320674460664
4	1.442320674460744	1.442320674460664	1.442320674460664	1.442320674460664
5	1.442320674460664	-	-	-
6	1.442320674460664	-	-	-

Example 5.3. Let $X = L^2[0, 1]$ and consider the operator equation (4), where

$$F(x)(t) = x(t) + \lambda \cos\left(\int_0^t x(s)ds\right) \text{ for all } x \in L^2[0, 1] \text{ and for all } t \in [0, 1],$$

where $\lambda \in \mathbb{R}$. Then, F is nowhere Fréchet differentiable but everywhere Gâteaux differentiable and $x \to F'_x$ is hemicontinuous. The Gâteaux derivative of F is given by

$$F'_x h(t) = h(t) + \lambda \int_0^t h(s)ds \sin\left(\int_0^t h(s)ds\right)$$

for all $h \in L^2[0,1]$ and for all $t \in [0,1]$.
For $x_0 = 0$, we have

$$F'_{x_0} = I.$$

Clearly, the operator F'_{x_0} is invertible. Again, we have

$$\|F'^{-1}_{x_0} F(x_0)\| = \lambda.$$

Let $r = 0.02$. For $x \in B_r[x_0]$ and using definition of norm, we can write

$$\begin{aligned} \|F'^{-1}_{x_0}(F'_x - F'_{x_0})h(t)\| &= |\lambda| \left\| \int_0^t h(s)ds \sin\left(\int_0^t x(s)ds\right) \right\| \\ &\leq |\lambda| \left[\int_0^1 \left| \int_0^t h(s)ds \sin\left(\int_0^t x(s)ds\right) \right|^2 \right]^{\frac{1}{2}} \\ &\leq |\lambda| \|h\|. \end{aligned}$$

Hence, we get

$$\|F'^{-1}_{x_0}(F'_x - F'_{x_0})\| \leq |\lambda|.$$

For $\lambda = 0.01$, $\eta = 0.01$ and for $\kappa = 0.01$, we have

$$\frac{\eta}{1 - 3\kappa} = 0.010309278350515 < r.$$

Hence all the assumptions of Theorem 3.2 are satisfied. Therefore, there exist a unique solution $x^* \in B_r[x_0]$ of the operator equation (4) such that the sequence $\{x_n\}$ defined by Algorithm 3.1 converges to x^*. Since the operator is not Fréchet differentiable, we can not apply Theorem 1.2 and hence Theorem 3.2 is a significant improvement of Theorem 1.2. Further, for $\alpha = 0.5$, it should be noted that $d_0 = 0.0222$ and $d = 0.0102$. Since $d < d_0$, hence the rate of convergence of SIP of Newton-like $\{x_n\}$ defined by Algorithm 3.1 is faster than that of Newton method $\{w_n\}$ defined by 5. Therefore, Theorem 3.2 is also significant improvement of previous results.

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