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Modified Newton Kantorovich Theorem for Common Solution of a System of Nonlinear Operator Equations

D. R. Sahu, Vipin Kumar Singh

Department of Mathematics, Faculty of Science
Banaras Hindu University, Varanasi 221005, India

Abstract. The Newton- Kantorovich hypothesis has been used by many authors as a sufficient condition for the convergence of Newton method to solve an operator equation of the form $F(x) = 0$. Motivated by normal S-iteration process, we introduce S-iteration process of Newton-like for finding a common solution of a system of nonlinear operator equations in Banach space setting. The results presented in this paper mainly improve the corresponding ones announced by many others and it is shown that rate of convergence of our algorithm is faster than modified Newton like method. Some numerical example are discussed in support of our result.

1. Introduction

Determining the roots of nonlinear operator equation has attracted the attention of pure and applied mathematicians for many years. A lot of problems about finding the solution of nonlinear equations are brought forward in many sciences and engineering. Mostly problems are formulate in terms of finding roots of the nonlinear operator equation

$$F(x) = 0. \tag{1.1}$$

The root of nonlinear operator equation (1.1) may not be generally expressed in simple way. So, in order to solve nonlinear equation, we often use iterative methods. There are many iterative method are available in literature. One of

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Email addresses: drsahudr@gmail.com (D. R. Sahu), vipinkumarsingh666@gmail.com (Vipin Kumar Singh)

the most popular method for solving such problems is Newton-Kantorovich method given by

$$x_{n+1} = x_n - {F'_{x_n}}^{-1}F(x_n)\ , n \in \mathbb{N}_0, \tag{1.2}$$

where F is a Fréchet differentiable operator at each point of an open convex subset D of Banach space X with values in another Banach space Y. There are numerous generalizations of Newton method for solving nonlinear operator equation (1.1) (see, for example, [3, 4, 6–9, 15]).

In method (1.2), the value of inverse of derivative is required at each iteration. This bring us a natural question how to modify iteration process (1.2) so that the computation of the inverse of derivative at each step can be avoid. Bartle [5], Dennis [6] and Rheinboldt [9] discussed the following modified Newton-Kantorovich method

$$x_{n+1} = x_n - {F'_{x_0}}^{-1}F(x_n), n \in \mathbb{N}_0. \tag{1.3}$$

This method converges only linearly but avoids the computation of expensive or impossible to compute inverse of F' except x_0.

Theorem 1.1. *[3, Theorem 2.2.7] Let F be a Fréchet differentiable operator defined on an open convex subset D of a Banach space X with values in a Banach space Y. For some $x_0 \in D$, let ${F'_{x_0}}^{-1} \in B(Y, X)$. Assume that ${F'_{x_0}}^{-1}$ and F satisfy the following conditions:*

(i) $\left\|F'^{-1}_{x_0}\right\| < \beta$, *for some* $\beta > 0$,

(ii) $\left\|F'^{-1}_{x_0}F(x_0)\right\| \leq \eta$, *for some* $\eta > 0$,

(iii) $\|F'_x - F'_y\| \leq k\|x - y\|$, *for all* $x, y \in D$ *and for some* $k > 0$,

(iv) $h = \eta\beta k \leq \frac{1}{2}$,

(v) $r = \frac{1-\sqrt{1-2h}}{h}\eta$,

and $B_r[x_0] \subseteq D$. Then the sequence $\{x_n\}$ generated by (1.3) is converges to a solution $x^* \in B_r[x_0]$ of (1.1).

In 2007, Agarwal et al. [1, 2] have introduced the S-iteration process as follows:

Let X be a normed space, D a nonempty convex subset of X and $A : D \to D$ an operator. Then, for arbitrary $x_0 \in D$, the S-iteration process is defined by

$$\begin{cases} x_{n+1} = (1 - \alpha_n)Ax_n + \alpha_n Ay_n, \\ y_n = (1 - \beta_n)x_n + \beta_n Ax_n,\ n = 0, 1, 2, ..., \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ with satisfying the condition

$$\sum_{n=0}^{\infty} \alpha_n\beta_n(1 - \beta_n) = \infty.$$

In [11], the first author introduced a normal S-iteration process as follows

$$x_{n+1} = A[(1 - \alpha_n)x_n + \alpha_n A(x_n)] \text{ for all } n \in \mathbb{N}_0. \tag{1.4}$$

In the following theorem, we compare the rate of convergence of iteration process (1.4), Picard iteration process and Mann iteration process for contraction mappings and show that iteration process (1.4) is superior than other contraction mappings.

Theorem 1.2. *[12, Theorem 3.1] Let C be a nonempty convex subset of a Banach space X. Let $F : C \to C$ be a contraction mapping with contraction constant k with fixed point x^*. Let $\{\alpha_n\}$ be a real sequence in $(0,1)$. For given x_0, y_0 and $z_0 \in C$ the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ generated by Picard, Mann and normal S-iteration process as follows:*

- *Picard Iteration Process :* $x_{n+1} = F(x_n)$ *for all* $n \in \mathbb{N}_0$.
- *Mann Iteration Process :* $y_{n+1} = (1-\alpha_n)y_n + \alpha_n F(y_n)$ *for all* $n \in \mathbb{N}_0$.
- *Normal S-iteration process:* $z_{n+1} = F[(1-\alpha_n)z_n + \alpha_n F(z_n)]$ $n \in \mathbb{N}_0$.

Then we have the following:

- $\|x_{n+1} - x^*\| \leq k\|x_n - x^*\|$ *for all* $n \in \mathbb{N}_0$.
- $\|y_{n+1} - x^*\| \leq [1-(1-k)\alpha_n]\|y_n - x^*\|$ *for all* $n \in \mathbb{N}_0$.
- $\|z_{n+1} - x^*\| \leq k[1-(1-k)\alpha_n]\|z_n - x^*\|$ *for all* $n \in \mathbb{N}_0$.

It is very clear that

$$0 < k[1-(1-k)\alpha_n] \leq [1-(1-k)\alpha_n] \leq k < 1$$

Clearly, the normal S-iteration process is faster than the Picard and Mann iteration processes for contraction mappings.

Motivated by normal S-iteration process, a new iteration process of Newton-like method (in real space) for finding the solution of equation $f(x) = 0$ has been introduced in [10]. The S-iteration process of Newton-like ([10]) is given by

$$\begin{cases} x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}, \\ z_n = (1-\alpha_n)x_n + \alpha_n y_n, \\ y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \end{cases} \tag{1.5}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$. Motivated by S-iteration process of Newton-like (1.5), an iteration process of modified Newton-like method for finding solution of operator equation (1.1) in Banach space setting is introduced in [13] as follows

$$\begin{cases} x_{n+1} = z_n - {F'_{x_0}}^{-1}F(z_n), \\ z_n = (1-\alpha)x_n + \alpha y_n, \\ y_n = x_n - {F'_{x_0}}^{-1}F(x_n), \end{cases} \tag{1.6}$$

where $\alpha \in (0,1)$.

Assume operators $F_i : D \subset X \to Y, (i = 1, 2, 3, ..., m)$, are Frêchet differentiable operators at each point of D. Let

$$\begin{cases} F_1(x) = 0, \\ F_2(x) = 0, \\ ..., \\ F_m(x) = 0 \end{cases} \tag{1.7}$$

be a system of nonlinear operator equations, where m is fixed positive integer. A point $x^* \in X$ is said to be common solution of (1.7) if $F_i(x^*) = 0, i = 1, 2, \cdots, m$.

Now our concern is the following:

Question 1.1. Is it possible to develop an iteration process to compute common solution of system (1.7)?

The purpose of this work is to design an iteration process which allows for approximating a common solution of (1.7) in Banach space setting and it provides an affirmative answers of above Question. Motivated by S-iteration process we propose the following iteration process for approximating to common solution of (1.7).

$$\begin{cases} \begin{cases} x_{n+1} = z_n^{(1)} - (F'_{1,x_0})^{-1} F_1(z_n^{(1)}), \\ z_n^{(1)} = (1-\alpha)x_n^{(2)} + \alpha y_n^{(1)}, \\ y_n^{(1)} = x_n^{(2)} - (F'_{1,x_0})^{-1} F_1(x_n^{(2)}), \end{cases} \\ \begin{cases} x_n^{(2)} = z_n^{(2)} - (F'_{2,x_0})^{-1} F_2(z_n^{(2)}), \\ z_n^{(2)} = (1-\alpha)x_n^{(3)} + \alpha y_n^{(2)}, \\ y_n^{(2)} = x_n^{(3)} - (F'_{2,x_0})^{-1} F_2(x_n^{(3)}), \end{cases} \\ \cdots, \\ \begin{cases} x_n^{(m)} = z_n^{(m)} - (F'_{m,x_0})^{-1} F_m(z_n^{(m)}), \\ z_n^{(m)} = (1-\alpha)x_n + \alpha y_n^{(m)}, \\ y_n^{(m)} = x_n - (F'_{m,x_0})^{-1} F_m(x_n), \end{cases} \end{cases} \tag{1.8}$$

where $\alpha \in (0, 1)$.

2. Preliminaries

Definition 2.1. [14] Let X *and* Y be normed spaces and let $F : X \to Y$ be a possibly nonlinear operator. Suppose that D be an open convex subset of X. If there exists a linear operator $F'_x \in B(X, Y)$ at a vector $x \in D$ such that

$$\lim_{\|h\| \to 0} \frac{\|F(x+h) - F(x) - F'_x h\|}{\|h\|} = 0,$$

for all vector $h \in X$. The operator F'_x is called the Fréchet derivative of operator F at vector x and the operator $F' : X \to B(X, Y)$, which assigns a continuous linear operator F'_x to a vector x, is known as the Fréchet derivative of F.

Definition 2.2. [1, 2] Let C be a nonempty subset of a normed space X and $T : C \to C$ be an operator. The operator $A_\alpha : C \to C$ is said to be S-operator generated by $\alpha \in (0, 1)$ and T if

$$A_\alpha = T[(1-\alpha)I + \alpha T],$$

where I is the identity operator.

Lemma 2.1. *[2, Theorem 4.1.5] Let (X, d) be a complete metric space and $F : X \to X$ a contraction mapping with Lipschitz constant $k \in (0, 1)$. Then we have the following:*

(i) There exists a unique fixed point $x^* \in X$.

(ii) For arbitrary $x_0 \in X$, the Picard iteration process defined by

$$x_{n+1} = F(x_n), n \in \mathbb{N}_0$$

converges to x^*.

Lemma 2.2. *[14, Theorem 9.4.2] Let X be a normed space while Y be a Banach space. If $F : X \to Y$ is a continuously Fréchet differentiable operator, then*

$$F(x) - F(y) = \int_0^1 F_{y+t(x-y)}(x-y)dt,$$

for all vectors $x, y \in X$.

Lemma 2.3. *Let D be an open convex subset of a Banach space X and F be a Fréchet differentiable operator at each point of D with value in another Banach space Y such that F satisfy all the condition of Theorem 1.1 with (iv) holding as strict inequality. Define a map $A : B_r[x_0] \to X$ as*

$$Ax = x - F'_{x_0}{}^{-1}F(x). \tag{2.1}$$

Then A is contraction self operator on $B_r[x_0]$ with contraction constant γ, where $\gamma = \eta r k$.

Proof. Set $\gamma = \beta r k$. Note that $\gamma = 1 - \sqrt{1-2h} < 1$ for $x, y \in B_r[x_0]$. Then, we have

$$\begin{aligned} Ax - Ay &= x - y - F'^{-1}_{x_0}F(x) + F'^{-1}_{x_0}F(y) \\ &= F'^{-1}_{x_0}[F'_{x_0}(x-y) - (F(x) - F(y))]. \end{aligned}$$

Consequently, we have

$$\begin{aligned} \|Ax - Ay\| &\leq \|F'^{-1}_{x_0}\| \int_0^1 \left\| F'_{x+t(x-y)}(x-y) - F'_{x_0}(x-y) \right\| dt \\ &\leq \beta k \int_0^1 \left\| x + t(x-y) - x_0 \right\| \|x-y\| dt. \\ &\leq \beta r k \|x-y\| \\ &\leq \gamma \|x-y\|, \end{aligned}$$

where $0 < \gamma = \beta rk < 1$. Thus, mapping A is contraction mapping. Now for $x \in B_r[x_0]$, we have

$$\begin{aligned} \|Ax - x_0\| &\leq \|Ax - x_0\| + \|Ax_0 - x_0\| \\ &\leq \|F'^{-1}_{x_0}\| \int_0^1 \left\|(F'_{x_0} - F'_{x_0+t(x-x_0)})(x - x_0)dt\right\| + \eta \\ &\leq \frac{\beta k}{2}\|x - x_0\|^2 + \eta \\ &\leq \frac{\beta r^2 k}{2} + \eta \\ &= r. \end{aligned}$$

Therefore, $Ax \in B_r[x_0]$. Hence by Lemma 2.1, operator A has unique fixed point in $B_r[x_0]$ □

Lemma 2.4. *Let D be an open convex subset of a Banach space X and F be continuously Fréchet differentiable operator at each point of D with value in another Banach space Y such that F satisfy all the condition of Lemma 2.3. Define a map $A_\alpha : B_r[x_0] \to X$ as*

$$A_\alpha(x) = A[(1 - \alpha)x + \alpha A(x)], \tag{2.2}$$

where $0 < \alpha < 1$. Then A_α is contraction map with contraction constant $0 < \gamma_\alpha = \gamma[(1 - \alpha) + \alpha\gamma] < 1$. Moreover, the sequence generated by

$$x_{n+1} = x_n - {F'_{x_0}}^{-1}[\alpha F(x_n) + F(x_n - \alpha {F'_{x_0}}^{-1} F(x_n))],\ n \in \mathbb{N}_0$$

converges to the solution x^* of equation (1.1).

Proof. It can be easily proved by using Lemma 2.1 and Lemma 2.3. □

3. Main Result

Theorem 3.1. *Let D be an open convex subset of a Banach space X and F_i ($i = 1, 2, \cdots, m$), be continuously Fréchet differentiable operators at each point of D with value in another Banach space Y such that $x_0 \in B_r[x_0]$, $(F'_{i,x_0})^{-1} \in B(Y, X)$ exists. For each $i = 1, 2, \cdots, m$, assume that F_i and $(F'_{i,x_0})^{-1}$ are satisfying the following conditions:*

(i) $\|(F'_{i,x_0})^{-1}\| \leq \beta_i$, *for some* $\beta_i > 0$;
(ii) $\|(F'_{i,x_0})^{-1} F_i(x_0)\| \leq \eta_i$, *for some* $\eta_i > 0$;
(iii) $\|F'_{i,x} - F'_{i,x_0}\| \leq k_i \|x - x_0\|$, *for all* $x \in D$ *and for some* $k_i > 0$.

Denote

(a) $k = max\{k_1, k_2, k_3..., k_m\}$;
(b) $\beta = max\{\beta_1, \beta_2, \beta_3, ..., \beta_m\}$;
(c) $\eta = max\{\eta_1, \eta_2, \eta_3, ..., \eta_m\}$.

Assume $B_r[x_0] = \{x \in X : \|x - x_0\| \leq r\} \subseteq D$, where $r = \frac{1-\sqrt{1-2h}}{h}\eta$, $h = \beta\eta k < \frac{1}{2}$. Let $x^* \in B_r[x_0]$ be the common solution of system (1.7). Then, the sequence generated by (1.8) converges to the unique solution $x^* \in B_r[x_0]$. Moreover, the following error estimate holds:

$$\|x_{n+1} - x^*\| \leq \gamma^{m(n+1)}\|x_0 - x^*\|.$$

Proof. Define a map $A_i : B_r[x_0] \to X$ for each $i = 1, 2, \cdots, m$ by

$$A_i(x) = x - (F'_{i,x_0})^{-1}F_i(x) \text{ and } S(x) = A_1A_2...A_3(x) \text{ for all } x \in B_r[x_0].$$

Lemma 2.3 implies that $S : B_r[x_0] \to B_r[x_0]$ is a contraction mapping with contraction constant γ^m. Since x^* is the solution of system (1.7), it follows that $x^* = A_i(x^*)$ for all $i = 1, 2, 3, .., m$ and the error estimate is

$$\|x_{n+1} - x^*\| = \|S(x_n) - S(x^*)\| \leq \gamma^m\|x_n - x^*\| \leq ... \leq \gamma^{m(n+1)}\|x_0 - x^*\|.$$

Hence the result. □

Theorem 3.2. *Let $A_i : B_r[x_0] \to X$ be operator same as in Theorem 3.1. Define an operator $S_\alpha : B_r[x_0] \to X$ as*

$$S_\alpha(x) = S[(1 - \alpha)x + \alpha S(x)], \text{ for all } n \in \mathbb{N}_0,$$

where $\alpha \in (0, 1)$. Then the sequence $\{x_n\}$ generated by

$$x_{n+1} = S_\alpha(x_n), \text{ for all } n \in \mathbb{N}_0$$

converges to x^* which is common solution of (1.7) with error estimate

$$\|x_{n+1} - x^*\| \leq (\gamma^m[(1 - \alpha) + \alpha\gamma^m])^{n+1}\|x^* - x_0\|. \tag{3.1}$$

Proof. Since S is contraction operator with contraction constant γ^m, we have

$$\|S_\alpha(x) - S_\alpha(y)\| \leq \gamma^m[(1 - \alpha) + \alpha\gamma^m]\|x - y\|, \text{ for all } x, y \in B_r[x_0]. \tag{3.2}$$

Note that $0 < \gamma^m[(1 - \alpha) + \alpha\gamma^m] < 1$. Therefore, S_α is self contraction map on $B_r[x_0]$. By Lemma 2.1, it has a fixed point in $B_r[x_0]$ which is common solution of (1.7). Using (3.2), one can obtained error estimate (3.1). □

Corollary 3.1. *Let D be an open convex subset of a Banach space X and F be continuously Fréchet differentiable operators at each point of D with value in another Banach space Y such that $x_0 \in B_r[x_0]$, ${F'_{x_0}}^{-1} \in B(Y, X)$ exists. Assume that F and ${F'_{x_0}}^{-1}$ are satisfying the following conditions:*

(i) $\|{F'_{x_0}}^{-1}\| \leq \beta$, *for some* $\beta > 0$;
(ii) $\|{F'_{x_0}}^{-1}F(x_0)\| \leq \eta$, *for some* $\eta > 0$;
(iii) $\|F'_x - F'_{x_0}\| \leq k\|x - x_0\|$, *for all* $x \in D$ *and for some* $k > 0$.

Assume that $h = \eta\beta K_0 < \frac{1}{2}$ and $B_r[x_0] \subseteq D$, where $r = \frac{1-\sqrt{1-2h}}{h}\eta$. Then, for fixed $\alpha \in (0, 1)$, we have the following:

(a) The operator $A : B_r[x_0] \to X$ defined by (2.1) is a contraction self-operator on $B_r[x_0]$ with Lipschitz constant βrk and the operator equation (1.1) has unique solution in $B_r[x_0]$.
(b) The S-operator $A_\alpha : B_r[x_0] \to X$ generated by α and A is a contraction self-operator on $B_r[x_0]$ with Lipschitz constant $\beta rk\,(1 - \alpha + \alpha\beta rk)$.

4. Numerical Example

In the following example, we use the norm $\|A\| = \sqrt{(\sum_{i=1}^{2} a_{ij}^2)}$, where A is a matrix of order $m \times n$ over the real field.

Example 4.1. Let $X = Y = \mathbb{R}^2$, $F_1, F_2 : D \subset \mathbb{R}^2 \to \mathbb{R}^2$ be an operator defined by

$$F_1(x, y) = (x^2 + y^2 - 13, 4x - 3y + 1)$$

and

$$F_2(x, y) = (x^2 - 6y + 14, 4x + y^2 - 17).$$

For $(x^0, y^0) = (1.5, 2.5)$ and $\alpha = \frac{1}{2}$, let

$$\beta_1 = \|(F'_{1,x_0})^{-1}\| = 0.264867094754090, \eta_1 = \|(F'_{1,x_0})^{-1}F_1(x_0)\| = 0.792541029969738, k_1 = 2$$

and

$$\beta_2 = \|(F'_{2,x_0})^{-1}\| = 0.237785089628095, \eta_2 = \|(F'_{2,x_0})^{-1}F_1(x_0)\| = 0.754397583073066, k_2 = 2.$$

Denote

$$\begin{cases} \beta = \max\{\beta_1, \beta_2\} = 0.264867094754090, \\ \eta = \max\{\eta_1, \eta_2\} = 0.792541029969738, \\ k = \max\{k_1, k_2\} = 2. \end{cases}$$

$$h = \beta\eta k = 0.419836080162997, r = \frac{1 - \sqrt{(1 - 2h)}}{h}\eta = 1.131870326642527.$$

The iteration process (1.8) can be written as

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} u_n^{(1)} \\ v_n^{(1)} \end{bmatrix} - \frac{1}{29}\begin{bmatrix} 3 & 5 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} \left(u_n^{(1)}\right)^2 + \left(v_n^{(1)}\right)^2 - 13 \\ 4u_n^{(1)} - 3v_n^{(1)} + 1 \end{bmatrix},$$

$$\begin{bmatrix} u_n^{(1)} \\ v_n^{(1)} \end{bmatrix} = \begin{bmatrix} (1-\alpha)x_n^{(2)} + \alpha x_n^{(1)} \\ (1-\alpha)y_n^{(2)} + \alpha y_n^{(1)} \end{bmatrix},$$

$$\begin{bmatrix} x_n^{(1)} \\ y_n^{(1)} \end{bmatrix} = \begin{bmatrix} x_n^{(2)} \\ y_n^{(2)} \end{bmatrix} - \frac{1}{29}\begin{bmatrix} 3 & 5 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} \left(x_n^{(2)}\right)^2 + \left(y_n^{(2)}\right)^2 - 13 \\ 4x_n^{(2)} - 3y_n^{(2)} + 1 \end{bmatrix},$$

$$\begin{bmatrix} x_n^{(2)} \\ y_n^{(2)} \end{bmatrix} = \begin{bmatrix} u_n^{(2)} \\ v_n^{(2)} \end{bmatrix} - \frac{1}{39}\begin{bmatrix} 5 & 6 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} \left(u_n^{(2)}\right)^2 - 6v_n^{(2)} + 14 \\ 4u_n^{(2)} + \left(v_n^{(2)}\right)^2 - 17 \end{bmatrix},$$

$$\begin{bmatrix} u_n^{(2)} \\ v_n^{(2)} \end{bmatrix} = \begin{bmatrix} (1-\alpha)x_n + \alpha x_n^{(3)} \\ (1-\alpha)y_n + \alpha y_n^{(3)} \end{bmatrix},$$

$$\begin{bmatrix} x_n^{(3)} \\ y_n^{(3)} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} - \frac{1}{39}\begin{bmatrix} 5 & 6 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} (x_n)^2 - 6y_n + 14 \\ 4x_n + (y_n)^2 - 17 \end{bmatrix}.$$

Following tables shows that the iteration process (1.8) is faster than the other two iteration processes (1.3) and (1.6).

Table I- Common solution of system of equations $F_1(x, y) = 0$ and $F_2(x, y) = 0$ by iteration process (1.8).

n	x_n	y_n
1	1.5	2.5
2	1.999895371030805	2.999860494707739
3	1.999999984046198	2.999999978728263
4	1.999999999997568	2.999999999996757
5	2.000000000000000	2.999999999999999
6	2.000000000000000	3.000000000000000

Table II- Common solution of system of equations $F_1(x, y) = 0$ and $F_2(u, v) = 0$ by iteration process (1.3).

n	x_n	y_n	u_n	v_n
1	1.5	2.5	1.5	2.5
2	2.051724137931035	3.068965517241379	2.070512820512820	2.993589743589744
3	1.986746074049777	2.982328098733035	1.991302323033092	3.008231974578128
4	1.999237109176559	3.004198326147545	1.999828505051537	2.998477247962054
5	1.999237109176559	2.998982812235411	2.000255895571149	3.000099370195917
6	2.000183978818759	3.000245305091679	1.999951895238703	3.000018607794967
7	1.999955581593114	2.999940775457485	2.000003304189458	2.999993635020191
8	2.000010721117468	3.000014294823291	2.000000555605998	3.000000828503061
9	1.999997412111030	2.999996549481373	1.999999801306307	2.999999993254205
10	2.000000624660931	3.000000832881240	2.000000026511360	2.999999980140071
11	1.999999849219663	2.999999798959551	1.999999999656481	3.000000004246801
12	2.000000036395247	3.000000048526996	1.999999999390687	2.999999999638090
13	1.999999991214940	2.999999988286587	2.000000000133795	2.999999999965346
14	2.000000002120532	3.000000002827376	1.999999999988178	3.000000000016388
15	1.99999999948814	2.999999999317530	1.999999999998994	2.999999999997527
16	2.000000000123551	3.000000000164734	2.000000000000509	3.000000000000087
17	1.999999999970177	2.999999999960237	1.999999999999922	3.000000000000046
18	2.000000000007198	3.000000000009598	2.000000000000003	2.999999999999989
19	1.999999999998263	2.999999999997683	2.000000000000001	3.000000000000001
20	2.000000000000419	3.000000000000559	1.999999999999999	3.000000000000000
21	1.999999999999899	2.999999999999865	2.000000000000000	3.000000000000000
22	2.000000000000024	3.000000000000032	2.000000000000000	3.000000000000000
23	1.999999999999994	2.999999999999992	2.000000000000000	3.000000000000000
24	2.000000000000001	3.000000000000002	2.000000000000000	3.000000000000000
25	2.000000000000000	3.000000000000000	2.000000000000000	3.000000000000000

Table III- Common solution of system of equations $F_1(x, y) = 0$ and $F_2(u, v) = 0$ by iteration process (1.6).

n	x_n	y_n	u_n	v_n
1	1.5	2.5	1.5	2.5
2	2.035479621960720	3.047306162614293	2.050710196142889	2.997250301336840
3	1.996744570750723	2.995659427667630	1.996871875931100	3.002209468491058
4	2.000297987822264	3.000397317096352	2.000064061424540	2.999776672637478
5	1.999972716338662	2.999963621784882	2.000009568677684	3.000012301200992
6	2.000002498023485	3.000003330697981	1.999998637619539	2.999999856261225
7	1.999999771286750	2.999999695049000	2.000000095672940	2.999999950702953
8	2.000000020940452	3.000000027920602	1.999999996912788	3.000000006039006
9	1.999999998082741	2.999999997443655	1.999999999827634	2.999999999612109
10	2.000000000175540	3.000000000234053	2.000000000034710	3.000000000009806
11	1.999999999983928	2.999999999978571	1.999999999997187	3.000000000000989
12	2.000000000001472	3.000000000001962	2.000000000000119	2.999999999999842
13	1.999999999999865	2.999999999999820	2.000000000000002	3.000000000000012
14	2.000000000000012	3.000000000000016	1.999999999999999	3.000000000000000
15	1.999999999999999	2.999999999999999	2.000000000000000	3.000000000000000
16	2.000000000000000	3.000000000000000	2.000000000000000	3.000000000000000

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