J Optim Theory Appl (2017) 172:102–127
DOI 10.1007/s10957-016-1031-x

Semilocal Convergence Analysis of *S*-iteration Process of Newton–Kantorovich Like in Banach Spaces

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Received: 4 December 2015 / Accepted: 18 October 2016 / Published online: 27 October 2016

Abstract In the present article, we establish a semilocal convergence theorem for the S-iteration process of Newton–Kantorovich like in Banach space setting for solving nonlinear operator equations and discuss its semilocal convergence analysis. We apply our result to solve the Fredholm-integral equations.

Keywords Nonlinear operator equation · Fréchet derivative · Newton's method · S-iteration process

Mathematics Subject Classification 65J15 · 47J25 · 65H10

1 Introduction

In the recent trend, a challenging problem is to obtain an approximate solution of a nonlinear equation that arises in many kind of engineering applications and scientific

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computation such as in civil engineering, mechanics, chemical engineering, numerical optimization, integral and differential equations, Game theory, Economics and in various branches of applied mathematics (see [1–4]).

A number of iterative methods to obtain a solution of nonlinear equations are available in the literature (see [3–7]). Newton's method is probably the well-known iterative method for finding approximate solution of these equations (see [5–12]). Newton's method is attractive because it converges quadratically under some reasonable hypothesis. Initially, a Russian mathematician Kantorovich [3] introduced the Newton's method in Banach space setting and studied its convergence analysis under some reasonable conditions. A number of authors used various ideas to prove local and semilocal convergence of Newton's method (see [1,2,7,11–22]) since Newton–Kantorovich method [3]. Most of them hit a certain boundedness condition of Kantorovich [3] and used the majorant principle (see [3,10]) to analyse the semilocal convergence of various Newton-like methods. In recent years, various modifications of Newton's method has been proposed in the literature which have either equal or better performance than the Newton's method (see [20–22]).

In [20], Ezquerro, González and Hernández have discussed the convergence analysis of Newton's method under a slightly more general condition as taken by Kantorovich [3]. In [21,22], Ezquerro, González and Hernández have further discussed the semilocal convergence analysis of Newton's method for m-times differentiable operator.

On the other hand, Agarwal, O'Regan and Sahu [23] have introduced the S-iteration process in Banach spaces for computing fixed points for nonlinear operators. It is remarked that the S-iteration process is independent of the Mann [24] and Ishikawa [25] iteration processes. The S-iteration process has a wide range of applications, including fixed point of a class of nonexpansive-type mappings in Banach spaces.

In [26], motivated by S-iteration process, the first author has introduced the normal S-iteration process. Note that the normal S-iteration process is applicable for finding solutions of constrained minimization problems and split feasibility problems (see [26]). Following [26, Theorem 3.6], we remark that the normal S-iteration process is faster than the Picard and Mann iteration processes for contraction mappings.

Motivated by normal S-iteration process, a new iteration process of Newton-like method for finding root of real-valued functions has been introduced in [27]. The semilocal convergence analysis of normal S-iteration process for modified Newton's method has been discussed in [18] and for generalized Newton's method in [28].

The purpose of this paper is to deal with semilocal convergence analysis for S-iteration process of Newton–Kantorovich like in Banach space setting for solving a nonlinear operator equation involving more than two times continuously Fréchet differentiable operator.

The remaining part of the paper is organized as follows: In Sects. 2 and 3, we give some basic definitions and some known results related to the main results of this paper. In Sect. 4, motivated by Sahu [27], we propose S-iteration process of Newton–Kantorovich like in a Banach space and give its semilocal convergence analysis. Our main result extends the main result of [22, Theorem 9] from Newton's method to S-iteration process of Newton–Kantorovich like. In Sect. 5, we give local convergence analysis of Newton–Kantorovich method and our S-iteration process of Newton–

Kantorovich like in a Banach space. Finally, we conclude Sect. 6 with numerical examples, where Kantorovich theorem [3, Theorem 6, p. 708] fails and our result holds good.

2 Newton Methods

For solving the nonlinear operator equation of the form

$$F(x) = 0, \tag{1}$$

Kantorovich (see [3]) introduced the following Newton's method in Banach space setting:

$$w_{n+1} = w_n - F_{w_n}'^{-1} F(w_n), n \in \mathbb{N}_0, \tag{2}$$

where $F : D \subset X \to Y$ is a nonlinear operator defined on an open convex subset D of a Banach space X with values into a Banach space Y and F_x' denotes the Fréchet derivative of F at point $x \in D$. Kantorovich [3] discussed the convergence analysis of method (2) under the following reasonable condition:

$$\|F_x''\| \leq K \text{ for all } x \in D \text{ and for some } K > 0. \tag{3}$$

In [20], Ezquerro et al. have studied the convergence analysis of Newton's method (2) under the following generalization of condition (3):

$$\|F_x''\| \leq \omega(\|x\|) \text{ for all } x \in D,$$

where $\omega : [0, \infty[\to \mathbb{R}$ is a continuous nondecreasing function. Recently, Ezquerro et al. [22] have discussed a semilocal convergence of (2) under the following generalized conditions:

(C1) $\|F_{x_0}'^{-1}\| \leq \frac{-1}{f'(t_0)}$ and $\|F_{x_0}'^{-1} F(x_0)\| \leq \frac{-f(t_0)}{f'(t_0)}$;

(C2) $\|F_{x_0}^{(j)}\| \leq f^{(j)}(t_0)$ for $j = 2, 3, \ldots, m-1$ and $\|F_x^{(m)}\| \leq f^{(m)}(t)$ for all $x \in D$ and $t \in [t_0, c]$ with $\|x - x_0\| \leq t - t_0$,

where $f \in C^{(m)}([t_0, c])$, $t_0, c \in \mathbb{R}$.

Let X be a normed space, D a nonempty convex subset of X and $T : D \to D$ an operator. Then, for arbitrary $x_0 \in D$, the normal S-iteration process [26] is defined by

$$x_{n+1} = T((1-\alpha_n)x_n + \alpha_n T x_n), n \in \mathbb{N}_0,$$

where $\{\alpha_n\}$ is a sequence in $]0, 1[$.

In [27], a new iteration process of Newton-like method for finding the solution of real equation $f(t) = 0$ has been introduced as follows:

$$t_{n+1} = s_n - \frac{f(s_n)}{f'(s_n)}, s_n = (1-\alpha_n)t_n + \alpha_n u_n, u_n = t_n - \frac{f(t_n)}{f'(t_n)}, n \in \mathbb{N}_0, \tag{4}$$

where $\{\alpha_n\}$ is a real sequence in $]0, 1[$.

3 Preliminaries

Let X and Y be two Banach spaces. Throughout the paper, we denote $B(X, Y)$ a collection of bounded linear operators from a Banach space X into another Banach space Y. For some $r > 0$, $B_r[x]$ and $B_r(x)$ are the closed and open balls with center x and radius r respectively, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $C^{(n)}([a, b])$ a collection of all n time continuously differentiable real-valued functions defined on $[a, b] \subset \mathbb{R}$.

In this section, we give some basic definitions and lemmas which are useful in the remaining part of the paper.

Definition 3.1 (*m*th *Fréchet derivative*) [6] Let D be a nonempty open subset of a Banach space X and Y be another Banach space. The operator $F : D \to Y$ is said to be m-times Fréchet differentiable at point $x \in D$ if there exits a m-linear bounded operator $F_x^{(m)} : X^m \to Y$ such that

$$\lim_{\|h_m\|\to 0} \frac{\|F_{(x+h_m)}^{(m-1)}(h_1, \ldots, h_{m-1}) - F_x^{(m-1)}(h_1, \ldots, h_{m-1}) - F_x^{(m)}(h_m, h_1, \ldots, h_{m-1})\|}{\|h_m\|} = 0,$$

where $h_i \in X$, $1 \leq i \leq m$.

Lemma 3.1 (Taylor's Theorem) [6] *Let X and Y be two Banach spaces and D be an open convex subset of X. Let $F : D \to Y$ be an m times continuously Fréchet differentiable operator. Then for all $x, y \in D$, we have*

$$F(y) = F(x) + \sum_{j=1}^{m-1} \frac{1}{j!} F_x^{(j)}(y - x)^j + \int_0^1 \frac{(1 - s)^{m-1}}{(m - 1)!} F_{x+s(y-x)}^{(m)}(y - x)^m ds \quad (5)$$

where $F_x^{(j)}(y - x)^j = F_x^{(j)}((y - x), (y - x), \ldots j$ times$), 1 \leq j \leq m$.

Definition 3.2 [9] Let $\{t_n\}$ be a sequence in $\mathbb{R}$ and $\{x_n\}$ a sequence in a Banach space X. Then, sequence $\{t_n\}$ is a majorizing sequence of $\{x_n\}$ if $\|x_n - x_{n-1}\| \leq t_n - t_{n-1}$ for all $n \in \mathbb{N}$.

Lemma 3.2 [9] *Let X be a Banach space and $\{x_n\}$ a sequence in X. Let $\{t_n\}$ be a majorizing sequence of $\{x_n\}$. Then, we have the following:*

(a) *Sequence $\{t_n\}$ is nondecreasing.*
(b) *If sequence $\{t_n\}$ converges to $t^* < \infty$, there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$ and $\|x^* - x_n\| \leq t^* - t_n$, for all $n \in \mathbb{N}_0$.*

Lemma 3.3 (Rall [5], p. 50) *Let L be a bounded linear operator on a Banach space X. Then, L^{-1} exists if and only if there is a bounded linear operator M in X such that M^{-1} exists, and $\|M - L\| < \frac{1}{\|M^{-1}\|}$. If L^{-1} exists, then*

$$\left\|L^{-1}\right\| \leq \frac{\left\|M^{-1}\right\|}{1 - \left\|1 - M^{-1}L\right\|} \leq \frac{\left\|M^{-1}\right\|}{1 - \left\|M^{-1}\right\| \|M - L\|}.$$

Lemma 3.4 ([22]) *Let $a, b \in \mathbb{R}$ and $f \in C^{(m)}([a, b]), m \geq 3$ be such that $f(a) > 0$, $f'(a) < 0$ and $f^{(j)}(a) > 0$ for $j = 2, 3, 4, \ldots, m-1$ and $f^{(m)}(t) > 0$ for all $t \in [a, b]$. Then, we have the following:*

(i) *If there exists $s^* \in]a, b[$ such that $f'(s^*) = 0$, then f is decreasing in $[a, s^*[$ and s^* is the unique minimum of f in $[a, b]$.*
(ii) *If $f(s^*) \leq 0$, then the real equation $f(t) = 0$ has at least one solution in $[a, b]$. Moreover, if t^* is the smallest solution of $f(t) = 0$ in $]a, b[$, then $a < t^* \leq s^*$.*

4 Semilocal Convergence Analysis

In this section, we extend S-iteration process of Newton-like (4) in Banach space setting and study its semilocal convergence analysis.

Algorithm 4.1 Let X and Y be two Banach spaces, D a nonempty open convex subset of X. Let $F : D \to Y$ be a nonlinear Fréchet differentiable operator. Let $\{\alpha_n\}$ be a sequence in $]0, 1[$. Starting with $x_0 \in D$ and after $x_n \in D$ defined, we define the next iterate x_{n+1} as follows:

$$y_n = x_n - F_{x_n}'^{-1} F(x_n),\ z_n = (1-\alpha_n)x_n + \alpha_n y_n,\ x_{n+1} = z_n - F_{z_n}'^{-1} F(z_n),\ n \in \mathbb{N}_0. \quad (6)$$

Algorithm 4.2 Let $\alpha_n = \alpha \in]0, 1[$. Starting with $x_0 \in D$ and after $x_n \in D$ is defined, we define the next iterate x_{n+1} as follows:

$$y_n = x_n - F_{x_n}'^{-1} F(x_n),\ z_n = (1-\alpha)x_n + \alpha y_n,\ x_{n+1} = z_n - F_{z_n}'^{-1} F(z_n),\ n \in \mathbb{N}_0, \quad (7)$$

where F and D are same as in Algorithm 4.1.

We begin with the following:

Lemma 4.1 *Let $a, b \in \mathbb{R}$ and $f : [a, b] \to \mathbb{R}$ be three times continuously differentiable function such that $f(a) > 0$, $f'(a) < 0$, $f''(a) > 0$ and $f'''(t) > 0$ for all $t \in [a, b]$. Then, we have the following:*

(i) *If there exists $s^* \in]a, b[$ such that $f'(s^*) = 0$, then f is decreasing in $[a, s^*[$ and s^* is the unique minimum of f in $[a, b]$.*
(ii) *If $f(s^*) \leq 0$, then the real equation $f(t) = 0$ has at least one solution in $[a, b]$. Moreover, if t^* is the smallest solution of $f(t) = 0$ in $]a, b[$, then $a < t^* \leq s^*$.*
(iii) *The sequences $\{t_n\}$ and $\{s_n\}$ generated by (4) are increasing and converge to t^*. Moreover,*

$$a \leq t_n \leq s_n \leq t_{n+1} < t^* \text{ for all } n \in \mathbb{N}_0.$$

Proof For the choice $m = 3$, parts (i) and (ii) follows from Lemma 3.4.

(iii) With the help of our assumptions and Taylor series, we can easily see that

$$f''(t) = f''(a) + f'''(a + \theta(t-a))(t-a) > 0 \text{ for all } t \in [a, b], \text{ where } \theta \in]0, 1[.$$

Let $\nu \in]0,1[$ and define $h_\nu, z_\nu : [a, s^*[\to \mathbb{R}$ by

$$h_\nu(t) = z_\nu(t) - \frac{f(z_\nu(t))}{f'(z_\nu(t))},\ z_\nu(t) = t - \nu \frac{f(t)}{f'(t)}. \tag{8}$$

For $t \in [a, t^*[$, we have

$$\begin{aligned} z'_\nu(t) &= 1 - \nu \frac{(f'(t))^2 - f(t)f''(t)}{(f'(t))^2} \\ &= \frac{(1-\nu)(f'(t))^2 + \nu f(t) f''(t)}{(f'(t))^2} > 0. \end{aligned}$$

Hence $z_\nu(t)$ is strictly increasing on $[a, t^*[$. Thus, we have $a < z_\nu(t) < z_\nu(t^*) = t^*$ for all $t \in [a, t^*[$. One can easily observe that $h_\nu(t)$ is increasing on $[a, t^*[$ with $a < h_\nu(t) < h_\nu(t^*) = t^*$ for all $t \in [a, t^*[$. From (8), we have

$$\begin{aligned} h_\nu(t) - z_\nu(t) &= -\frac{f(z_\nu(t))}{f'(z_\nu(t))} \\ &= -\frac{1}{f'(z_\nu(t))}\left[f(z_\nu(t)) - f(t) - f'(t)(z_\nu(t) - t) + (1-\nu)f(t)\right] \\ &= -\frac{1}{f'(z_\nu(t))}\left[\int_0^1 f''(t + s(z_\nu(t) - t))(1-s)(z_\nu(t) - t)^2 \mathrm{d}s + (1-\nu)f(t)\right] > 0 \end{aligned} \tag{9}$$

for all $t \in [a, t^*[$. From (8), we have

$$z_\nu(t) = t - \nu \frac{f(t)}{f'(t)} \geq t \text{ for all } t \in [a, t^*]. \tag{10}$$

Therefore, from (9) and (10), we obtain

$$t \leq z_\nu(t) \leq h_\nu(t) \leq t^* \ \text{ for all } t \in [a, t^*]$$

and hence

$$a \leq t_n \leq s_n = z_{\alpha_n}(t_n) \leq h_{\alpha_n}(t_n) = t_{n+1} \leq t^* \text{ for all } n \in \mathbb{N}_0,$$

where $\{\alpha_n\}$ is a sequence in $]0,1[$. This completes the proof. □

We show that the sequence generated by (4) has quadratic convergence as the sequence generated by (11):

Theorem 4.1 *Let $a, b \in \mathbb{R}$ and $f : [a,b] \to \mathbb{R}$ be twice continuously differentiable function. Assume that $t^* \in]a,b[$ a simple root of $f(t) = 0$ and t_0 is sufficiently close* *to t^*. Let $\{t_n\}$ be a real sequence generated by S-iteration process of Newton-like (4) and a real sequence $\{r_n\}$ generated by Newton method*

$$r_{n+1} = r_n - \frac{f(r_n)}{f'(r_n)}, n \in \mathbb{N}_0. \tag{11}$$

If both the sequences $\{r_n\}$ and $\{t_n\}$ with $\alpha_n = \alpha \in]0, 1[$ converges to t^* starting from t_0, then we have the following error estimates:

$$\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2}\frac{f''(t^*)}{f'(t^*)} + o(\varepsilon_n^2),$$

where $\varepsilon_n = r_n - t^*$ for all $n \in \mathbb{N}_0$ and

$$\varepsilon'_{n+1} = (1-\alpha)^2\left(\frac{1}{2}\frac{f''(t^*)}{f'(t^*)}\varepsilon'^2_n + o(\varepsilon'^2_n)\right),$$

where $\varepsilon'_n = t_n - t^*$ for all $n \in \mathbb{N}_0$.

Proof Note that t^* is a solution of the equation $f(t) = 0$. Set $r_n = t^* + \varepsilon_n$ for all $n \in \mathbb{N}_0$. Using the Taylor expansions, we have

$$\begin{aligned}
\varepsilon_{n+1} &= \varepsilon_n - \frac{f(t^* + \varepsilon_n)}{f'(t^* + \varepsilon_n)}\\
&= \varepsilon_n - \frac{\varepsilon_n f'(t^*) + \frac{\varepsilon_n{}^2}{2} f''(t^*) + o(\varepsilon_n{}^2)}{f'(t^*) + \varepsilon_n f''(t^*) + o(\varepsilon_n{}^2)}\\
&= \varepsilon_n - \left(\varepsilon_n + \frac{\varepsilon_n{}^2}{2}\frac{f''(t^*)}{f'(t^*)} + o(\varepsilon_n{}^2)\right)\left(1 + \varepsilon_n\frac{f''(t^*)}{f'(t^*)} + o(\varepsilon_n{}^2)\right)^{-1}\\
&= \frac{\varepsilon_n{}^2}{2}\frac{f''(t^*)}{f'(t^*)} + o(\varepsilon_n{}^2).
\end{aligned}$$

Set $s_n = t^* + \eta_n$ and $t_n = t^* + \varepsilon'_n$ and using the Taylor expansions, we have

$$\begin{aligned}
\eta_n &= \varepsilon'_n - \alpha\frac{f(t^* + \varepsilon'_n)}{f'(t^* + \varepsilon'_n)}\\
&= \varepsilon'_n - \alpha\frac{\varepsilon'_n f'(t^*) + \frac{\varepsilon'^2_n}{2} f''(t^*) + o(\varepsilon'^2_n)}{f'(t^*) + \varepsilon'_n f''(t^*) + o(\varepsilon'^2_n)}\\
&= \varepsilon'_n - \alpha\left[\varepsilon'_n + \frac{\varepsilon'^2_n}{2}\frac{f''(t^*)}{f'(t^*)} + o(\varepsilon'^2_n)\right]\left[1 + \varepsilon'_n\frac{f''(t^*)}{f'(t^*)} + o(\varepsilon'^2_n)\right]^{-1}\\
&= (1-\alpha)\varepsilon'_n + \alpha\frac{\varepsilon'^2_n}{2}\frac{f''(t^*)}{f'(t^*)} + o(\varepsilon'^2_n)
\end{aligned} \tag{12}$$

and

$$\begin{aligned}\varepsilon'_{n+1} &= \eta_n - \frac{f(t^*+\eta_n)}{f'(t^*+\eta_n)}\\ &= \eta_n - \frac{\eta_n f'(t^*) + \frac{\eta_n^2}{2} f''(t^*) + o(\eta^2)}{f'(t^*) + \eta_n f''(t^*) + o(\eta_n^2)}\\ &= \eta_n - \left(\eta_n + \frac{\eta_n^2}{2}\frac{f''(t^*)}{f'(t^*)} + o(\eta_n^2)\right)\left(1 + \eta_n \frac{f''(t^*)}{f'(t^*)} + o(\eta_n^2)\right)^{-1}.\end{aligned} \tag{13}$$

From (12) and (13), we have

$$\varepsilon'_{n+1} = (1-\alpha)^2\left(\frac{1}{2}\frac{f''(t^*)}{f'(t^*)}\varepsilon_n'^2 + o(\varepsilon_n'^2)\right).$$

This completes the proof. □

We now ready to establish our main results:

Theorem 4.2 *Let X and Y be two Banach spaces, D a nonempty open convex subset of X and $x_0 \in D$. Let $F : D \to Y$ be a nonlinear $m(m \geq 3)$ times continuously Fréchet differentiable operator such that $F_{x_0}'^{-1} \in B(Y, X)$ exists. Let $t_0, c \in \mathbb{R}$ and let $f \in C^{(m)}([t_0, c])$ such that $f(t_0) > 0$, $f'(t_0) \leq -1$, $f^{(j)}(t_0) > 0$ for $j = 2, 3, \ldots, m-1$ and $f^{(m)}(t) > 0$ for all $t \in [t_0, c]$. Let $s^* \in]t_0, c[$ be the solution of $f'(t) = 0$ such that $f(s^*) \leq 0$. Suppose that $t^* \in]t_0, c[$ is the smallest positive root of $f(t) = 0$ and $B_r[x_0] \subset D$, where $r = t^* - t_0$. Assume that the conditions $(C1)-(C2)$ hold. Then, we have the following:*

(a) *The sequences $\{x_n\}$ and $\{z_n\}$ generated by (6) are well defined, remain in $B_r[x_0]$ with the following estimates:*
  (i) $\|z_{n-1} - x_{n-1}\| \leq s_{n-1} - t_{n-1}$ *and* $\|F_{z_{n-1}}'^{-1}\| \leq -\frac{1}{f'(s_{n-1})}$ *for all* $n \in \mathbb{N}$;
  (ii) $\|x_n - z_{n-1}\| \leq t_n - s_{n-1}$ *and* $\|x_n - x_{n-1}\| \leq t_n - t_{n-1}$ *for all* $n \in \mathbb{N}$;
  (iii) $\|F_{x_n}'^{-1}\| \leq -\frac{1}{f'(t_n)}$, $\|F_{z_{n-1}}^{(i)}\| \leq f^{(i)}(s_{n-1})$ *and* $\|F(x_n)\| \leq f(t_n)$ *for all* $n \in \mathbb{N}$, $i = 2, 3, \ldots m-1$;
  (iv) $\|F_{x_n}^{(i)}\| \leq f^{(i)}(t_n)$ *for all* $n \in \mathbb{N}$ *and* $i = 2, 3, \ldots, m-1$,
  where $\{t_n\}$ and $\{s_n\}$ are real sequences generated by (4).

(b) *The sequence $\{x_n\}$ converges to the solution $x^* \in B_r[x_0]$ of operator equation (1) and the following error estimate holds:*

$$\|x^* - x_n\| \leq t^* - t_n \text{ for all } n \in \mathbb{N}_0.$$

(c) *If there exists another solution $t^{**} \in (t_0, c)$ of $f(t) = 0$ such that $t^* < t^{**}$, x^* is the unique solution of (1) in $B_{t^{**}-t_0}(x_0) \cap D$ and if $t^* = t^{**}$, x^* is unique in $B_r[x_0] \cap D$.*

Proof (a) Note that $f'(t)$ is strictly increasing on $[t_0, s^*)$ and $f'(t) < 0$ for all $t \in [t_0, s^*[$. Then for $t, s \in [t_0, s^*[$ with $t < s$, we have

$$\frac{f'(s)}{f'(t)} < 1. \tag{14}$$

Let $x \in B_r[x_0]$, $t \in [t_0, c]$ be such that $F_x'^{-1}$, $\frac{1}{f'(t)}$ exist, $y = x - F_x'^{-1}F(x) \in B_r[x_0]$, $s = t - \frac{f(t)}{f'(t)} \in [t_0, c]$ with $\|y - x\| \leq s - t$, $\|x - x_0\| \leq t - t_0$ and if $\|F_x'^{-1}\| \leq -\frac{1}{f'(t)}$, $\|F_x^{(i)}\| \leq f^{(i)}(t)$, $i = 2, 3, \ldots, m$. Then, we have

$$\begin{aligned}
\|F(y)\| &= \left\| F(x) + \sum_{j=1}^{m-1} \frac{F_x^{(j)}(y-x)^j}{j!} + \frac{1}{(m-1)!}\int_0^1 (1-\theta)^{m-1} F_{x+\theta(y-x)}^{(m)}(y-x)^m \mathrm{d}\theta \right\| \\
&= \left\| \sum_{j=2}^{m-1} \frac{F_x^{(j)}(y-x)^j}{j!} + \frac{1}{(m-1)!}\int_0^1 (1-\theta)^{m-1} F_{x+\theta(y-x)}^{(m)}(y-x)^m \mathrm{d}\theta \right\| \\
&\leq \sum_{j=2}^{m-1} \frac{\|F_x^{(j)}\|\|y-x\|^j}{j!} + \frac{1}{(m-1)!}\int_0^1 (1-\theta)^{m-1} \|F_{x+\theta(y-x)}^{(m)}\|\|y-x\|^m \mathrm{d}\theta \\
&\leq \sum_{j=2}^{m-1} \frac{f^{(j)}(t)(s-t)^j}{j!} + \frac{1}{(m-1)!}\int_0^1 (1-\theta)^{m-1} f^{(m)}(t+\theta(s-t))(s-t)^m \mathrm{d}\theta \\
&= f(t) + \sum_{j=1}^{m-1} \frac{f^{(j)}(t)(s-t)^j}{j!} \\
&\quad + \frac{1}{(m-1)!}\int_0^1 (1-\theta)^{m-1} f^{(m)}(t+\theta(s-t))(s-t)^m \mathrm{d}\theta = f(s)
\end{aligned} \tag{15}$$

and

$$\begin{aligned}
&\|I - F_x'^{-1}F_y'\| \\
&= \left\| I - F_x'^{-1}\left(F_x' + \sum_{j=2}^{m-1} \frac{F_x^{(j)}(y-x)^{j-1}}{(j-1)!} + \frac{1}{(m-2)!}\int_0^1 F_{x+\theta(y-x)}^{(m)}(1-\theta)^{m-2}(y-x)^{m-1}\mathrm{d}\theta \right) \right\| \\
&= \left\| -F_x'^{-1}\left(\sum_{j=2}^{m-1} \frac{F_x^{(j)}(y-x)^{j-1}}{(j-1)!} + \frac{1}{(m-2)!}\int_0^1 F_{x+\theta(y-x)}^{(m)}(1-\theta)^{m-2}(y-x)^{m-1}\mathrm{d}\theta \right) \right\| \\
&\leq \|F_x'^{-1}\|\left(\sum_{j=2}^{m-1} \frac{\|F_x^{(j)}\|}{(j-1)!}\|y-x\|^{j-1} + \frac{1}{(m-2)!}\int_0^1 \|F_{x+\theta(y-x)}^{(m)}\|(1-\theta)^{m-2}\|y-x\|^{m-1}\mathrm{d}\theta \right) \\
&\leq -\frac{1}{f'(t)}\left(\sum_{j=2}^{m-1} \frac{f^{(j)}(t)}{(j-1)!}(s-t)^{j-1} + \frac{1}{(m-2)!}\int_0^1 f^{(m)}(t+\theta(s-t))(1-\theta)^{m-2}(s-t)^{m-1}\mathrm{d}\theta \right) \\
&= 1 - \frac{1}{f'(t)}\left(\sum_{j=1}^{m-1} \frac{f^{(j)}(t)}{(j-1)!}(s-t)^{j-1} + \frac{1}{(m-2)!}\int_0^1 f^{(m)}(t+\theta(s-t))(1-\theta)^{m-2}(s-t)^{m-1}\mathrm{d}\theta \right)
\end{aligned}$$

$$= 1 - \frac{f'(s)}{f'(t)}. \tag{16}$$

We now proceed with the following steps:
Step I *Sequences* $\{x_n\}$ *and* $\{z_n\}$ *are well defined, remain in* $B_r[x_0]$ *and estimates* (i)–(iv) *are true for* $n = 1$.

From (4) and (6), we have

$$\|z_0 - x_0\| \leq \alpha_0 \|F_{x_0}'^{-1} F(x_0)\| \leq -\alpha_0 \frac{f(t_0)}{f'(t_0)} = s_0 - t_0 \leq r.$$

Thus, $z_0 \in B_r[x_0]$. By (14) and (16), we obtain

$$\|I - F_{x_0}'^{-1} F_{z_0}'\| \leq 1 - \frac{f'(s_0)}{f'(t_0)} < 1.$$

Therefore, by Lemma 3.3, we have

$$\|F_{z_0}'^{-1}\| \leq -\frac{1}{f'(s_0)}. \tag{17}$$

Noticing that $f'(t_0) \leq -1$ and $\|z_0 - x_0\| \leq s_0 - t_0$. Using (4) and (6), we have

$$\begin{aligned}
\|x_1 - z_0\| &= \|F_{z_0}'^{-1} F(z_0)\| \\
&= \|F_{z_0}'^{-1}(F(z_0) - F(x_0) - F_{x_0}'(y_0 - x_0))\| \\
&= \|F_{z_0}'^{-1}(F(z_0) - F(x_0) - F_{x_0}'(z_0 - x_0) - F_{x_0}'(y_0 - z_0))\| \\
&= \|F_{z_0}'^{-1}(F(z_0) - F(x_0) - F_{x_0}'(z_0 - x_0) + (1 - \alpha_0) F(x_0))\| \\
&\leq \|F_{z_0}'^{-1}\| \int_0^1 \|F''_{x_0 + s(z_0 - x_0)}\| (1 - s) \|z_0 - x_0\|^2 \mathrm{d}s \\
&\quad + (1 - \alpha_0) \|(F_{x_0}'^{-1} F_{z_0}')^{-1}\| \|F_{x_0}'^{-1} F(x_0)\| \\
&\leq -\frac{1}{f'(s_0)} \left(\int_0^1 f''(t_0 + s(s_0 - t_0))(1 - s)(s_0 - t_0)^2 \mathrm{d}s - (1 - \alpha_0) \frac{f(t_0)}{f'(t_0)} \right) \\
&= -\frac{1}{f'(s_0)} \left(f(s_0) - f(t_0) - f'(t_0)(s_0 - t_0) - (1 - \alpha_0) \frac{f(t_0)}{f'(t_0)} \right) \\
&= -\frac{1}{f'(s_0)} \left(f(s_0) - f(t_0) + \alpha_0 f(t_0) - (1 - \alpha_0) \frac{f(t_0)}{f'(t_0)} \right) \\
&= -\frac{1}{f'(s_0)} \left(f(s_0) - (1 - \alpha_0) f(t_0) \left(1 + \frac{1}{f'(t_0)} \right) \right) \\
&\leq -\frac{f(s_0)}{f'(s_0)} = t_1 - s_0.
\end{aligned}$$

Hence

$$\|x_1 - x_0\| \leq \|x_1 - z_0\| + \|z_0 - x_0\| \leq t_1 - t_0 \leq r.$$

Therefore, $x_1 \in B_r[x_0]$. Using (14) and (16), we have

$$\|I - F'^{-1}_{z_0} F'_{x_1}\| \le 1 - \frac{f'(t_1)}{f'(s_0)} < 1.$$

By Lemma 3.3 and (17), we obtain that $\|F'^{-1}_{x_1}\| \le -\frac{1}{f'(t_1)}$.

Now for $i = 2, 3, \ldots, m-1$, using Taylor's series (5), we have

$$\begin{aligned}
\|F^{(i)}_{z_0}\| &= \left\| F^{(i)}_{x_0} + \sum_{j=i+1}^{m-1} \frac{F^{(j)}_{x_0}(z_0 - x_0)^{j-i}}{(j-i)!} + \int_0^1 \frac{F^{(m)}_{x_0+t(z_0-x_0)}(1-t)^{m-i-1}(z_0-x_0)^{m-i}}{(m-i-1)!} \mathrm{d}t \right\| \\
&\le \left\| F^{(i)}_{x_0} \right\| + \sum_{j=i+1}^{m-1} \frac{\|F^{(j)}_{x_0}\| \|(z_0 - x_0)^{j-i}\|}{(j-i)!} + \int_0^1 \frac{\|F^{(m)}_{x_0+t(z_0-x_0)}\|}{(m-i-1)!}(1-t)^{m-i-1}\|z_0 - x_0\|^{m-i} \mathrm{d}t \\
&\le f^{(i)}(t_0) + \sum_{j=i+1}^{m-1} \frac{f^{(j)}(t_0)(s_0 - x_0)^{j-i}}{(j-i)!} + \int_0^1 \frac{f^{(m)}(t_0 + t(s_0 - t_0))}{(m-i-1)!}(1-t)^{m-i-1}(s_0 - t_0)^{m-i} \mathrm{d}t \\
&= f^{(i)}(s_0). \qquad (18)
\end{aligned}$$

From (15) and (18), we have $\|F(x_1)\| \le f(t_1)$. For $i = 2, 3, \ldots, m-1$, by Taylor's series (5), we have

$$\begin{aligned}
\|F^{(i)}_{x_1}\| &= \left\| F^{(i)}_{z_0} + \sum_{j=i+1}^{m-1} \frac{F^{(j)}_{z_0}(x_1 - z_0)^{j-i}}{(j-i)!} + \int_0^1 \frac{F^{(m)}_{z_0+t(x_1-z_0)}(1-t)^{m-i-1}(x_1-z_0)^{m-i}}{(m-i-1)!} \mathrm{d}t \right\| \\
&\le \left\| F^{(i)}_{z_0} \right\| + \sum_{j=i+1}^{m-1} \frac{\|F^{(j)}_{z_0}\| \|x_1 - z_0\|^{j-i}}{(j-i)!} + \int_0^1 \frac{\|F^{(m)}_{z_0+t(x_1-z_0)}\|}{(m-i-1)!}(1-t)^{m-i-1} \mathrm{d}t \|x_1 - z_0\|^{m-i} \\
&\le f^{(i)}(s_0) + \sum_{j=i+1}^{m-1} \frac{f^{(j)}(s_0)(t_1 - s_0)^{j-i}}{(j-i)!} + \int_0^1 \frac{f^{(m)}(s_0 + t(t_1 - s_0))}{(m-i-1)!}(1-t)^{m-i-1} \mathrm{d}t (t_1 - s_0)^{m-i} \\
&= f^{(i)}(t_1).
\end{aligned}$$

Step II *Sequences* $\{x_n\}$ *and* $\{z_n\}$ *are well defined, remain in* $B_r[x_0]$ *and estimates* (i)–(iv) *are true for all* $n > 1$.

Let $k > 1$ be a fixed integer. Assume that sequences $\{x_n\}$ and $\{z_n\}$ are well defined and remain in $B_r[x_0]$ and estimates (i)–(iv) are true for $n = 2, 3, \ldots, k$. From (4) and (6), we have

$$\begin{aligned}
\|z_k - x_k\| = \alpha_k \|F'^{-1}_{x_k} F(x_k)\| &\le \alpha_k \|F'^{-1}_{x_k}\| \|F(x_k)\| \\
&\le -\alpha_k \frac{f(t_k)}{f'(t_k)} = s_k - t_k
\end{aligned}$$

and

$$\|z_k - x_0\| \le \|z_k - x_{k-1}\| + \|x_{k-1} - x_0\| \le s_k - t_0 \le t^* - t_0 \le r.$$

Hence $z_k \in B_r[x_0]$. From (14) and (16), we get $\|I - F'^{-1}_{x_k} F'_{z_k}\| \le 1 - \frac{f'(s_k)}{f'(t_k)} < 1$. Therefore, by Lemma 3.3, we have $\|F'^{-1}_{z_k}\| \le -\frac{1}{f'(s_k)}$. Set $v_k = x_k + t(z_k - x_k)$, $w_k =$

$z_{k-1} + t(x_k - z_{k-1})$ and $b_k = t_k + t(s_k - t_k)$, $c_k = s_{k-1} + t(t_k - s_{k-1})$, $t \in]0, 1[$. By Taylor's series (5), (6) and (4), we have

$$
\begin{aligned}
F(z_k) &= F(z_k) - F(x_k) - F'_{x_k}(y_k - x_k)\\
&= F(z_k) - F(x_k) - F'_{x_k}(z_k - x_k) - F'_{x_k}(y_k - z_k)\\
&= F(z_k) - F(x_k) - F'_{x_k}(z_k - x_k)) + (1-\alpha_k)F(x_k)\\
&= F(z_k) - F(x_k) - F'_{x_k}(z_k - x_k) + (1-\alpha_k)(F(x_k) - F(z_{k-1}) - F'_{z_{k-1}}(x_k - z_{k-1}))\\
&= \int_0^1 F''_{x_k+t(z_k-x_k)}(1-t)(z_k - x_k)^2 \mathrm{d}t\\
&+(1-\alpha_k)\int_0^1 F''_{z_{k-1}+t(x_k-z_{k-1})}(1-t)(x_k - z_{k-1})^2 \mathrm{d}t\\
&= \int_0^1 F''_{v_k}(1-t)(z_k - x_k)^2 \mathrm{d}t + (1-\alpha_k)\int_0^1 F''_{w_k}(1-t)(x_k - z_{k-1})^2 \mathrm{d}t\\
&= \int_0^1 \left(\sum_{j=2}^{m-1} \frac{F^{(j)}_{x_0}(v_k - x_0)^{j-2}}{(j-2)!} + \int_0^1 \frac{F^{(m)}_{x_0+s(v_k-x_0)}(1-s)^{m-3}(v_k - x_0)^{m-2}}{(m-3)!}\mathrm{d}s\right)(1-t)(z_k - x_k)^2 \mathrm{d}t\\
&+(1-\alpha_k)\int_0^1 \left(\sum_{j=2}^{m-1} \frac{F^{(j)}_{x_0}(w_k - x_0)^{j-2}}{(j-2)!} + \int_0^1 \frac{F^{(m)}_{x_0+s(w_k-x_0)}(1-s)^{m-3}(w_k - x_0)^{m-2}}{(m-3)!}\mathrm{d}s\right)\\
&(1-t)(x_k - z_{k-1})^2 \mathrm{d}t
\end{aligned}
$$

and

$$
\begin{aligned}
\|F(z_k)\| \le\ & \int_0^1 \left(\sum_{j=2}^{m-1} \frac{\|F^{(j)}_{x_0}\|\|v_k - x_0\|^{j-2}}{(j-2)!} + \int_0^1 \frac{\|F^{(m)}_{x_0+s(v_k-x_0)}\|(1-s)^{m-3}\|v_k - x_0\|^{m-2}}{(m-3)!}\mathrm{d}s\right)(1-t)\mathrm{d}t\\
&\|z_k - x_k\|^2\\
&+(1-\alpha_k)\int_0^1 \left(\sum_{j=2}^{m-1} \frac{\|F^{(j)}_{x_0}\|\|w_k - x_0\|^{j-2}}{(j-2)!} + \int_0^1 \frac{\|F^{(m)}_{x_0+s(w_k-x_0)}\|(1-s)^{m-3}\|w_k - x_0\|^{m-2}}{(m-3)!}\mathrm{d}s\right)\\
&(1-t)\|x_k - z_{k-1}\|^2 \mathrm{d}t\\
\le\ & \int_0^1 \left(\sum_{j=2}^{m-1} \frac{f^{(j)}(t_0)(b_k - t_0)^{j-2}}{(j-2)!} + \int_0^1 \frac{f^{(m)}(t_0 + s(b_k - t_0))(1-s)^{m-3}(b_k - t_0)^{m-2}}{(m-3)!}\mathrm{d}s\right)\\
&(1-t)(s_k - t_k)^2 \mathrm{d}t + (1-\alpha_k)\int_0^1 \left(\sum_{j=2}^{m-1} \frac{f^{(j)}(t_0)(c_k - t_0)^{j-2}}{(j-2)!}\right.\\
&\left.+\int_0^1 \frac{f^{(m)}(t_0 + s(c_k - t_0))(1-s)^{m-3}(c_k - t_0)^{m-2}}{(m-3)!}\mathrm{d}s\right)(1-t)(t_k - s_{k-1})^2 \mathrm{d}t\\
=\ & \int_0^1 f''(t_k + t(s_k - t_k))(1-t)(s_k - t_k)^2 \mathrm{d}t\\
&+(1-\alpha_k)\int_0^1 f''(s_{k-1} + t(t_k - s_{k-1}))(1-t)(t_k - s_{k-1})^2 \mathrm{d}t\\
=\ & f(s_k) - f(t_k) - f'(t_k)(s_k - t_k) + (1-\alpha_k)(f(t_k) - f(s_{k-1}) - f'(s_{k-1})(t_k - s_{k-1}))\\
=\ & f(s_k) - f(t_k) + \alpha_k f(t_k) + (1-\alpha_k)(f(t_k) - f(s_{k-1}) - f'(s_{k-1})(t_k - s_{k-1}))
\end{aligned}
$$

$$= f(s_k) - (1-\alpha_k)f(t_k) + (1-\alpha_k)(f(t_k) - f(s_{k-1}) - f'(s_{k-1})(t_k - s_{k-1}))$$
$$= f(s_k).$$

From (4) and (6), we have

$$\|x_{k+1} - z_k\| \leq \|F_{z_k}^{\prime -1} F(z_k)\| \leq -\frac{f(s_k)}{f'(s_k)} = t_{k+1} - s_k$$

and

$$\|x_{k+1} - x_0\| \leq \|x_{k+1} - z_k\| + \|z_k - x_0\| \leq t_{k+1} - t_0 \leq t^* - t_0 = r.$$

Hence $x_{k+1} \in B_r[x_0]$. Using (14) and (16), we have

$$\|I - F_{z_k}^{\prime -1} F'_{x_{k+1}}\| \leq 1 - \frac{f'(t_{k+1})}{f'(s_k)} < 1.$$

Thus, by Lemma 3.3, we get

$$\|F_{x_{k+1}}^{\prime -1}\| \leq -\frac{1}{f'(t_{k+1})}.$$

For $i = 2, 3 \ldots, m-1$, by Taylor's series (5) and (14), we have

$$\begin{aligned}
\|F_{z_k}^{(i)}\| &= \left\| F_{x_k}^{(i)} + \sum_{j=i+1}^{m-1} \frac{F_{x_k}^{j}(z_k - x_k)^{j-i}}{(j-i)!} + \int_0^1 \frac{F_{x_k + t(z_k - x_k)}^{(m)}(1-t)^{m-i-1}(z_k - x_k)^{m-i}}{(m-i-1)!} \mathrm{d}t \right\| \\
&\leq \left\| F_{x_k}^{(i)} \right\| + \sum_{j=i+1}^{m-1} \frac{\|F_{x_k}^{(j)}\| \|z_k - x_k\|^{j-i}}{(j-i)!} \\
&\quad + \int_0^1 \frac{\|F_{x_k + t(z_k - x_k)}^{(m)}\|}{(m-i-1)!} (1-t)^{m-i-1} \|z_k - x_k\|^{m-i-1} \mathrm{d}t \\
&\leq f^{(i)}(t_k) + \sum_{j=i+1}^{m-1} \frac{f^{(j)}(t_k)(s_k - t_k)^{j-i}}{(j-i)!} \\
&\quad + \int_0^1 \frac{f^{(m)}(t_k + t(s_k - t_k))}{(m-i-1)!} (1-t)^{m-i-1} (s_k - t_k)^{m-i} \mathrm{d}t \\
&= f^{(i)}(s_k).
\end{aligned} \tag{19}$$

From (15) and (19), we have

$$\|F(x_{k+1})\| \leq f(t_{k+1}).$$

For $i = 2, 3, \ldots, m-1$, using Taylor's series (5), we have

$$
\begin{aligned}
\|F^{(i)}_{x_{k+1}}\| &= \Bigg\| F^{(i)}_{z_k} + \sum_{j=i+1}^{m-1} \frac{F^{(j)}_{z_k}(x_{k+1}-z_k)^{j-i}}{(j-i)!} \\
&\quad + \int_0^1 \frac{F^{(m)}_{z_k+t(x_{k+1}-z_k)}(1-t)^{m-i-1}(x_{k+1}-z_k)^{m-i}}{(m-i-1)!}\mathrm{d}t \Bigg\| \\
&\le \left\| F^{(i)}_{z_k} \right\| + \sum_{j=i+1}^{m-1} \frac{\|F^{(j)}_{z_k}\| \|x_{k+1}-z_k\|^{j-i}}{(j-i)!} \\
&\quad + \int_0^1 \frac{\|F^{(m)}_{z_k+t(x_{k+1}-z_k)}\|}{(m-i-1)!}(1-t)^{m-i-1}\|x_{k+1}-z_k\|^{m-i}\mathrm{d}t \\
&\le f^{(i)}(s_k) + \sum_{j=i+1}^{m-1} \frac{f^{(j)}(s_k)(t_{k+1}-s_k)^{j-i}}{(j-i)!} \\
&\quad + \int_0^1 \frac{f^{(m)}(s_k+t(t_{k+1}-s_k))}{(m-i-1)!}(1-t)^{m-i-1}(t_{k+1}-s_k)^{m-i}\mathrm{d}t \\
&= f^{(i)}(t_{k+1}).
\end{aligned}
$$

Hence, we conclude that sequences $\{x_n\}$ and $\{z_n\}$ are well defined and remain in $B_r[x_0]$ and the estimates $(i)-(iv)$ are true for $n = k+1$. Therefore, by mathematical induction part (a) is true for all $n \in \mathbb{N}_0$.

(b) By part (a), sequence $\{t_n\}$ is majorizing sequence of $\{x_n\}$. Since sequence $\{t_n\}$ is convergent, therefore, by Lemma 3.2, sequence $\{x_n\}$ is convergent. Let it converges to x^*. Using continuity of F, f and part (a)(iii), we have

$$\|F(x^*)\| = \lim_{n\to\infty} \|F(x_n)\| \le \lim_{n\to\infty} f(t_n) = f(t^*) = 0.$$

Thus, x^* is the solution of nonlinear equation (1). For $n \in \mathbb{N}_0$, $p \in \mathbb{N}$, one can easily observe that

$$\|x_{n+p} - x_n\| \le \sum_{j=n}^{n+p-1} \|x_{j+1} - x_j\| \le \sum_{j=n}^{n+p-1} (t_{j+1} - t_j) = t_{n+p} - t_n.$$

Taking limit $p \to \infty$, we have

$$\|x^* - x_n\| \le t^* - t_n \text{ for all } n \in \mathbb{N}_0$$

and hence $\|x^* - x_0\| \le t^* - t_0 = r$. Thus, $x^* \in B_r[x_0]$.

(c) Suppose that $t^* < t^{**}$ and y^* is another solution of $F(x) = 0$ in $B_{(t^{**}-t_0)}(x_0) \cap D$. Then,

$$\|y^* - x_0\| \le \sigma(t^{**} - t_0) \text{ with } \sigma \in]0, 1[.$$

Let $k > 1$ be a fixed integer. Assume that $\|y^* - x_n\| \le \sigma^{2^n}(t^{**} - t_n)$ is true for $n = 2, 3, \ldots, k$. Now from (6), we have

$$\begin{aligned}
\|y^* - y_k\| &= \|y^* - x_k + F_{x_k}^{\prime -1} F(x_k)\| \\
&= \|F_{x_k}^{\prime -1}(F(y^*) - F(x_k) - F'_{x_k}(y^* - x_k))\| \\
&\le \|F_{x_k}^{\prime -1}\| \int_0^1 \|F''_{x_k + t(y^* - x_k)}(1-t)(y^* - x_k)\|^2 \mathrm{d}t \\
&\le -\frac{1}{f'(t_k)} \int_0^1 f''(t_k + t(t^{**} - t_k))(1-t)\mathrm{d}t \|y^* - x_k\|^2 \\
&= -\frac{\mu_k}{f'(t_k)} \|y^* - x_k\|^2,
\end{aligned}$$

where $\mu_k = \int_0^1 f''(t_k + t(t^{**} - t_k))(1-t)\mathrm{d}t$. From (4), we have

$$\begin{aligned}
t^{**} - u_k &= t^{**} - t_k + \frac{f(t_k)}{f'(t_k)} \\
&= -\frac{1}{f'(t_k)}[f(t^{**}) - f(t_k) - f'(t_k)(t^{**} - t_k)] \\
&= -\frac{1}{f'(t_k)} \int_0^1 f''(t_k + t(t^{**} - t_k))(1-t)\mathrm{d}t(t^{**} - t_k)^2 \\
&= -\frac{\mu_k}{f'(t_k)}(t^{**} - t_k)^2.
\end{aligned}$$

Therefore,

$$\|y^* - y_k\| \le \frac{t^{**} - u_k}{(t^{**} - t_k)^2} \|y^* - x_k\|^2 \le \sigma^{2^{k+1}}(t^{**} - u_k).$$

Again, from (6), we have

$$\begin{aligned}
\|y^* - z_k\| &\le \|y^* - (1 - \alpha_k)x_k - \alpha_k y_k\| \\
&\le (1 - \alpha_k)\|y^* - x_k\| + \alpha_k \|y^* - y_k\| \\
&= (1 - \alpha_k)\sigma^{2^k}(t^{**} - t_k) + \alpha_k \sigma^{2^{k+1}}(t^{**} - u_k) \\
&\le \sigma^{2^k}[(1 - \alpha_k)(t^{**} - t_k) + \alpha_k(t^{**} - u_k)].
\end{aligned}$$

Now from (6), we have

$$\begin{aligned}
\|y^* - x_{k+1}\| &= \|y^* - z_k + F_{z_k}'^{-1}F(z_k)\| \\
&= \|F_{z_k}'^{-1}(F(y^*) - F(z_k) - F_{z_k}'(y^* - z_k))\| \\
&\leq \|F_{z_k}'^{-1}\| \int_0^1 \|F''_{z_k + t(y^* - z_k)}(1-t)(y^* - z_k)\|^2 \mathrm{d}t \\
&\leq -\frac{1}{f'(s_k)} \int_0^1 f''(s_k + t(t^{**} - s_k))(1-t)\mathrm{d}t \|y^* - z_k\|^2 \\
&= -\frac{\mu_k'}{f'(s_k)}\|y^* - z_k\|^2,
\end{aligned}$$

where $\mu_k' = \int_0^1 f''(s_k + t(t^{**} - s_k))(1-t)\mathrm{d}t$. From (4), we have

$$\begin{aligned}
t^{**} - t_{k+1} &= t^{**} - s_k + \frac{f(s_k)}{f'(s_k)} \\
&= -\frac{1}{f'(s_k)}(f(t^{**}) - f(s_k) - f'(s_k)(t^{**} - s_k)) \\
&= -\frac{1}{f'(s_k)} \int_0^1 f''(s_k + t(t^{**} - s_k))(1-t)\mathrm{d}t(t^{**} - s_k)^2 \\
&= -\frac{\mu_k'}{f'(s_k)}(t^{**} - s_k)^2 \\
&= -\frac{\mu_k'}{f'(s_k)}[t^{**} - (1-\alpha_k)t_k - \alpha_k u_k]^2 \\
&= -\frac{\mu_k'}{f'(s_k)}[(1-\alpha_k)(t^{**} - t_k) + \alpha_k(t^{**} - u_k)]^2.
\end{aligned}$$

Therefore,

$$\|y^* - x_{k+1}\| \leq \frac{(t^{**} - t_{k+1})\|y^* - z_k\|^2}{[(1-\alpha_k)(t^{**} - t_k) + \alpha_k(t^{**} - u_k)]^2} \leq \sigma^{2^{k+1}}(t^{**} - t_{k+1}).$$

Therefore, by mathematical induction $\|y^* - x_n\| \leq \sigma^{2^n}(t^{**} - t_n)$ is true for all $n \in \mathbb{N}_0$, so that $y^* = x^*$.

If $t^* = t^{**}$, corresponding result follows from previous one. □

Set $\alpha_n = \alpha \in]0, 1[$ in Theorem 4.2, we have:

Corollary 4.1 *Under the hypothesis of Theorem 4.2, we have the following:*

(a) *The sequences $\{x_n\}$ and $\{z_n\}$ generated by (7) are well defined, remain in $B_r[x_0]$.*
(b) *The sequence $\{x_n\}$ converges to the solution $x^* \in B_r[x_0]$ of operator equation (1) and the following error estimate holds:*

$$\|x^* - x_n\| \leq t^* - t_n \textit{ for all } n \in \mathbb{N}_0,$$

where sequence $\{t_n\}$ *generated by* (4) *with* $\alpha_n = \alpha \in]0, 1[$.

(c) *If there exists another solution* $t^{**} \in]t_0, c[$ *of* $f(t) = 0$ *such that* $t^* < t^{**}$, x^* *is the unique solution of* (1) *in* $B_{t^{**}-t_0}(x_0) \cap D$ *and if* $t^* = t^{**}$, x^* *is unique in* $B_r[x_0] \cap D$.

Now, we discuss the semilocal convergence analysis of S-iteration process of Newton-like (6) using an important majorizing function $f : [0, \frac{1}{\gamma}[\to \mathbb{R}$ defined by

$$f(t) = \eta - t + \frac{b\gamma t^2}{1 - \gamma t}, \tag{20}$$

where η, γ and b are positive real numbers. Function f was used by Wang in his work [29,30] for $b = 1$ in connection to the Smale's α-theory [31] in order to finding approximate zero. The function f was also used by Shen and Li [32], Argyros, Khattri and Hilout [33] to study the convergence analysis of inexact Newton method.

One can easily observe that $f'(t)$ is strictly increasing on $[0, \frac{1}{\gamma}[$ and it has unique positive zero

$$s^* = \left(1 - \sqrt{\frac{b}{b+1}}\right)\frac{1}{\gamma} \in \left]0, \frac{1}{\gamma}\right[\tag{21}$$

and if $\sigma = \gamma\eta \le 1 + 2b - 2\sqrt{b(b+1)}$, then equation $f(t) = 0$ has two real roots

$$t^* = \frac{1 + \sigma - \sqrt{(1+\sigma)^2 - 4(b+1)\sigma}}{2(b+1)\gamma} \text{ and } t^{**} = \frac{1 + \sigma + \sqrt{(1+\sigma)^2 - 4(b+1)\sigma}}{2(b+1)\gamma} \tag{22}$$

satisfying

$$\eta \le t^* \le s^* \le t^{**}.$$

Finally, we study the semilocal convergence analysis for Algorithm 4.1.

Theorem 4.3 *Let* η, γ *and* b *be positive real numbers. Let* X *and* Y *be two Banach spaces,* D *a nonempty open convex subset of* X *and* $x_0 \in D$. *Let* $F : D \to Y$ *be a nonlinear* $m(m \ge 3)$ *times continuously Fréchet differentiable operator such that* $F_{x_0}'^{-1} \in B(Y, X)$ *exists. Assume that* F *satisfies the conditions* $(C1) - (C2)$ *with a real-valued function* f *defined by* (20). *Assume that* $\sigma = \gamma\eta \le 1 + 2b - 2\sqrt{b(b+1)}$ *and* $B_{t^*}[x_0] \subset D$, *where* t^* *is defined by* (22). *Then, we have the following:*

(i) *The sequence* $\{x_n\}$ *generated by* (6) *is well defined and remains in* $B_{t^*}[x_0]$ *and converges to the solution* $x^* \in B_{t^*}[x_0]$ *of operator equation (1). Moreover, the following error estimate holds:*

$$\|x^* - x_n\| \le t^* - t_n, n \in \mathbb{N}_0,$$

where $\{t_n\}$ *is a real sequence generated by* (4).

(ii) *If there exists another solution* $t^{**} \in]0, \frac{1}{\gamma}[$ *of equation* $f(t) = 0$ *such that* $t^* < t^{**}$, x^* *is the unique solution of* (1) *in* $B_{t^{**}}(x_0) \cap D$.

Proof (i) One can easily observe that for $t_0 = 0$, $f(t_0) = \eta$, $f'(t_0) = -1$, $f^{(j)}(t_0) = bj!\gamma^{j-1} > 0$ for $j = 2, 3, \ldots, m-1$ and

$$f^{(m)}(t) = \frac{bm!\gamma^{m-1}}{(1-\gamma t)^{m+1}} > 0 \text{ for all } t \in [0, 1/\gamma[.$$

Thus the function f is m-times continuously differentiable on $]0, 1/\gamma[$ and $f(s^*) \leq 0$, where s^* is defined by (21). Now

$$\|F_{x_0}'^{-1}\| \leq -\frac{1}{f'(t_0)} = 1, \|F_{x_0}'^{-1}F(x_0)\| \leq -\frac{f(t_0)}{f'(t_0)} = \eta, \|F_{x_0}^{(j)}\| \leq f^{(j)}(t_0) = bj!\gamma^{j-1} > 0$$

for all $j = 2, 3, \ldots, m-1$ and $\|F_x^{(m)}\| \leq f^{(m)}(t)$ for all $t \in [0, 1/\gamma[$. Thus, all the assumption of Theorem 4.2 are satisfied. Therefore, part (a) follows from Theorem 4.2.

(ii) This part also follows from Theorem 4.2. □

Remark 4.1 (1) Theorem 9 of [22] deals with the convergence analysis of Newton's method (2). Our results deal with the approximate solution of operator equation (1) in the context of S-iteration process of Newton– Kantorovich like (4).

(2) Algorithm 4.1 is a generalization of (4) in infinite dimensional Banach spaces. Our result extends the [27, Theorem 6.3, p-114], in the frame work of infinite Banach spaces.

5 Local Convergence Analysis

In this section, we discuss the local convergence analysis of Newton method (2) and S-iteration process of Newton-like (6).

Lemma 5.1 *Let η and γ be positive real numbers such that $\gamma\eta \leq 3 - 2\sqrt{2}$ and define a function $f : [0, 1/\gamma[\to \mathbb{R}$ by*

$$f(t) = \eta - t + \frac{\gamma t^2}{1 - \gamma t}.$$

Let X and Y be two Banach spaces, D a nonempty open convex subset of X and $F : D \to Y$ a nonlinear operator such that F is a twice continuously Fréchet differentiable on D. Let $x^* \in D$ be a solution of (1) *and $F_{x^*}'^{-1}$ exists and let $B_\varrho(x^*) \subset D$, where $\varrho = \left(1 - \frac{1}{\sqrt{2}}\right)\frac{1}{\gamma}$. Assume that F satisfies the following condition:*

$$\|F_{x^*}'^{-1}F_x''\| \leq f''(\|x - x^*\|) \text{ for all } x \in B_\varrho(x^*).$$

Define Newton operator $G : B_\varrho(x^*) \to X$ by

$$G(x) = x - F_x'^{-1}F(x),\ x \in B_\varrho(x^*) \tag{23}$$

Then, we have the following:

(a) *The equation* $f(t) = 0$ *has two real roots* r^* *and* r^{**}*. They satisfy*

$$r^* = \frac{1+\gamma\eta - \sqrt{(1+\gamma\eta)^2 - 8\gamma\eta}}{4\gamma} \le \varrho \le r^{**} = \frac{1+\gamma\eta + \sqrt{(1+\gamma\eta)^2 - 8\gamma\eta}}{4\gamma}.$$

(b) $F_x'^{-1}$ *exists for all* $x \in B_\varrho(x^*)$ *and hence* G *is well defined on* $B_\varrho(x^*)$*.*
(c) *For all* $x \in B_\varrho(x^*)$*, we have*

$$\|F_x'^{-1} F_{x^*}'\| \le -\frac{1}{f'(\|x - x^*\|)} \tag{24}$$

and

$$\|G(x) - x^*\| \le -\frac{1}{f'(\|x - x^*\|)} \int_0^1 f''(s\|x - x^*\|)sds\|x - x^*\|^2. \tag{25}$$

Proof (a) It follows from [11].
(b) For $x \in B_\varrho(x^*)$, we have

$$\begin{aligned}
\|I - F_{x^*}'^{-1} F_x'\| &= \|F_{x^*}'^{-1}(F_{x^*}' - F_x')\| \\
&\le \int_0^1 \|F_{x^*}'^{-1} F''_{x+t(x^*-x)}\|\|x - x^*\|\mathrm{d}t \\
&\le \int_0^1 f''((1-t)\|x - x^*\|)\|x - x^*\|\mathrm{d}t \\
&= f'(\|x - x^*\|) - f'(0) < 1.
\end{aligned}$$

Therefore, by Banach Lemma 3.3, $F_x'^{-1}$ exists. This proves that the operator G is well defined on $B_\varrho(x^*)$.
(c) The inequality (24) follows from part (b). Let $x \in B_\varrho(x^*)$. Using (23) and (24), we have

$$\begin{aligned}
\|G(x) - x^*\| &= \|x - x^* - F_x'^{-1} F(x)\| \\
&= \|F_x'^{-1}(F(x^*) - F(x) - F_x'(x^* - x))\| \\
&\le \|F_x'^{-1} F_{x^*}'\| \int_0^1 \|F_{x^*}'^{-1} F''_{x+t(x^*-x)}\|(1-t)\|x - x^*\|^2\mathrm{d}t \\
&\le -\frac{1}{f'(\|x - x^*\|)} \int_0^1 f''((1-t)\|x - x^*\|)(1-t)\|x - x^*\|^2 \\
&= -\frac{1}{f'(\|x - x^*\|)} \int_0^1 f''(s\|x - x^*\|)s\mathrm{d}s\|x - x^*\|^2.
\end{aligned}$$

□

Theorem 5.1 *Suppose that all the assumptions of Lemma 5.1 hold. Let* $\kappa = -\frac{1}{f'(r^*)}\int_0^1 f''(sr^*)sds$ *and*

$$\theta := \kappa r^* < 1. \tag{26}$$

Let $x_0 = w_0 \in B_{r^*}(x^*)$. *Let* $\{w_n\}$ *be the sequence generated from* w_0 *and defined by Newton method (2) and let* $\{x_n\}$ *be the sequence generated from* x_0 *and defined by S-iteration process of Newton-like (6). Then, we have the following:*

(a) *The sequence* $\{w_n\}$ *is in* $B_{r^*}(x^*)$ *and it converges to* x^* *with the error bound:*

$$\|w_{n+1} - x^*\| \leq \kappa\|w_n - x^*\|^2, n \in \mathbb{N}_0. \tag{27}$$

(b) *The sequence* $\{x_n\}$ *is in* $B_{r^*}(x^*)$, *and it converges to* x^* *with the error bound:*

$$\|x_{n+1} - x^*\| \leq \kappa\,(1 - \alpha_n(1-\theta))^2\,\|x^* - x_n\|^2, n \in \mathbb{N}_0. \tag{28}$$

Proof (a) The Newton method (2) can be written as $w_{n+1} = G(w_n)$, where G is defined by (23). Using (25) and (26), we have

$$\begin{aligned}
\|w_1 - x^*\| &\leq -\frac{1}{f'(\|x_0 - x^*\|)}\int_0^1 f''(s\|x_0 - x^*\|)s\mathrm{d}s\|x_0 - x^*\|^2\\
&\leq -\frac{1}{f'(r^*)}\int_0^1 f''(sr^*)s\mathrm{d}s\|x_0 - x^*\|^2\\
&\leq \theta\|x_0 - x^*\| < r^*.
\end{aligned}$$

Thus, $w_1 \in B_{r^*}(x^*)$. Let $k > 1$ be a fixed positive integer. Suppose that $w_k \in B_{r^*}(x^*)$. From (25) and (26), we have

$$\begin{aligned}
\|w_{k+1} - x^*\| &\leq -\frac{1}{f'(\|w_k - x^*\|)}\int_0^1 f''(s\|w_k - x^*\|)s\mathrm{d}s\|w_k - x^*\|^2\\
&\leq -\frac{1}{f'(r^*)}\int_0^1 f''(sr^*)sds\|w_k - x^*\|^2\\
&\leq \kappa\|w_k - x^*\|^2\\
&< \theta\|w_k - x^*\| < r^*.
\end{aligned}$$

Hence $w_n \in B_{r^*}(x^*)$ for $n = k+1$. Therefore, by mathematical induction $w_n \in B_{r^*}(x^*)$ for all $n \in \mathbb{N}_0$.

Form (25) and (26), we get

$$\|w_{n+1} - x^*\| \leq -\frac{1}{f'(\|w_n - x^*\|)}\int_0^1 f''(s\|w_n - x^*\|)s\mathrm{d}s \leq \kappa\|w_n - x^*\|^2, n \in \mathbb{N}_0. \tag{29}$$

It follows from (26) that

$$\|w_{n+1} - x^*\| \leq \theta^{n+1}\|w_0 - x^*\|, n \in \mathbb{N}_0.$$

Therefore, the sequence $\{w_n\}$ converges to x^*.

(b) The S-iteration process of Newton-like (6) can be written as

$$y_n = G(x_n), z_n = (1-\alpha_n)x_n + \alpha_n y_n, x_{n+1} = G(z_n), n \in \mathbb{N}_0, \tag{30}$$

where G is defined by (23). Using (25) and (30), we have

$$\begin{aligned}
\|z_0 - x^*\| &\le (1-\alpha_0)\|x_0 - x^*\| + \alpha_0\|y_0 - x_0\| \\
&\le (1-\alpha_0)\|x_0 - x^*\| - \frac{\alpha_0}{f'(\|x_0 - x^*\|)}\int_0^1 f''(s\|x_0 - x^*\|)s\mathrm{d}s\|x_0 - x^*\|^2 \\
&\le (1-\alpha_0)\|x_0 - x^*\| - \frac{\alpha_0}{f'(r^*)}\int_0^1 f''(sr^*)s\mathrm{d}s\|x_0 - x^*\|^2 \\
&\le (1-\alpha_0(1-\kappa r^*))\|x_0 - x^*\| < r^*.
\end{aligned}$$

Thus, $z_0 \in B_{r^*}(x^*)$. Again, from (25), (26) and (30), we have

$$\begin{aligned}
\|x_1 - x^*\| &\le -\frac{1}{f'(\|z_0 - x^*\|)}\int_0^1 f''(s\|z_0 - x^*\|)s\mathrm{d}s\|z_0 - x^*\|^2 \\
&\le -\frac{1}{f'(r^*)}\int_0^1 f''(sr^*)s\mathrm{d}s\|z_0 - x^*\|^2 \\
&= \kappa(1-\alpha_0(1-\kappa r^*))^2\|x_0 - x^*\|^2 \\
&< \theta(1-\alpha_0(1-\theta))^2\|x_0 - x^*\| < r^*.
\end{aligned}$$

Thus, $x_1 \in B_{r^*}(x^*)$. Let $k > 1$ be a fixed positive integer. Suppose that $x_k \in B_{r^*}(x^*)$. From (25) and (30), we have

$$\begin{aligned}
\|z_k - x^*\| &\le (1-\alpha_k)\|x_k - x^*\| + \alpha_k\|y_k - x_k\| \\
&\le (1-\alpha_k)\|x_k - x^*\| - \frac{\alpha_k}{f'(\|x_k - x^*\|)}\int_0^1 f''(s\|x_k - x^*\|)s\mathrm{d}s\|x_k - x^*\|^2 \\
&\le (1-\alpha_k)\|x_k - x^*\| - \frac{\alpha_k}{f'(r^*)}\int_0^1 f''(sr^*)s\mathrm{d}s\|x_k - x^*\|^2 \\
&\le (1-\alpha_k(1-\kappa r^*))\|x_k - x^*\| < r^*.
\end{aligned}$$

Thus, $z_k \in B_{r^*}(x^*)$. Again, from (25), (26) and (30), we have

$$\begin{aligned}
\|x_{k+1} - x^*\| &\le -\frac{1}{f'(\|z_k - x^*\|)}\int_0^1 f''(s\|z_k - x^*\|)s\mathrm{d}s\|z_k - x^*\|^2 \\
&\le -\frac{1}{f'(r^*)}\int_0^1 f''(sr^*)s\mathrm{d}s\|z_k - x^*\|^2 \\
&= \kappa(1-\alpha_k(1-\kappa r^*))^2\|x_k - x^*\|^2 \\
&< \theta(1-\alpha_k(1-\theta))^2\|x_k - x^*\| < r^*.
\end{aligned}$$

Thus, $x_n \in B_{r^*}(x^*)$ for $n = k+1$. Therefore, by mathematical induction, $x_n \in B_{r^*}(x^*)$ for all $n \in \mathbb{N}_0$.

Form (25) and (26), we get

$$\|x_{n+1} - x^*\| \leq \kappa(1 - \alpha_k(1-\theta))^2 \|x_n - x^*\|^2, n \in \mathbb{N}_0.$$

Using (26), we have

$$\|x_{n+1} - x^*\| \leq \theta(1 - \alpha_n(1-\theta))^2 \|x_n - x^*\| \leq \theta^{n+1}\|x_0 - x^*\|, n \in \mathbb{N}_0.$$

Therefore, the sequence $\{x_n\}$ converges to x^*. □

Remark 5.1 One can observe from (27) and (28) that

$$\kappa\,(1 - \alpha_n(1-\theta))^2 < \kappa \text{ for all } n \in \mathbb{N}_0. \tag{31}$$

The strict inequality (31) shows that the error estimate in S-iteration process of Newton-like (6) is sharper than that of Newton method (2).

6 Numerical Examples

Example 6.1 Let $D = X = Y = C([0, 1])$ be the space of all continuous real-valued functions defined on $[0, 1]$ with norm $\|x\| = \max\{|x(s)| : s \in [0, 1]\}$. Consider the integral equation (see [34]) as follows:

$$x(s) = \frac{3}{10} + \frac{x^2(s)}{2} + \frac{1}{4}\int_0^1 \frac{s}{s+t} x^3(t)\mathrm{d}t. \tag{32}$$

Define $F : C([0, 1]) \to C([0, 1])$ by

$$F(x)(s) = x(s) - \frac{3}{10} - \frac{x^2(s)}{2} - \frac{1}{4}\int_0^1 \frac{s}{s+t} x^3(t)\mathrm{d}t.$$

The first, second and third Fréchet derivative of F at point $x \in D$ are

$$\begin{aligned}
F'_x y(s) &= y(s) - x(s)y(s) - \frac{3}{4}\int_0^1 \frac{s}{s+t} x^2(t)y(t)\mathrm{d}t,\\
F''_x(yz)(s) &= -y(s)z(s) - \frac{3}{2}\int_0^1 \frac{s}{s+t} x(t)y(t)z(t)\mathrm{d}t,\\
F'''_x(yzw)(s) &= -\frac{3}{2}\int_0^1 \frac{s}{s+t} y(t)z(t)w(t)\mathrm{d}t
\end{aligned}$$

for all $y, z, w \in C([0, 1])$. Observe that $\|F''_x\|$ is not bounded in $C([0, 1])$. For $x_0 = 0 \in D$, we have $F'_{x_0}h(s) = h(s)$ for all $h \in X$ which implies that $F'^{-1}_{x_0}U(s) = U(s)$

for all $U \in C([0,1])$ and $\|F_{x_0}^{\prime -1}\| = 1$, $\|F_{x_0}^{''}\| = 1$, $\|F_x^{'''}\| = \frac{3\ln 2}{2}$, $\|F_{x_0}^{\prime -1}F(x_0)\| \leq \frac{3}{10} = \eta$ and

$$\gamma = \max\left\{\frac{\|F_{x_0}^{''}\|}{2}, \left(\frac{\|F_{x_0}^{'''}\|}{6}\right)^{\frac{1}{2}}\right\} = \max\{0.5, 0.4163\} = 0.5.$$

Since $\gamma\eta = 0.15 < 3 - 2\sqrt{2}$, the function $f(t)$ defined by (20) for $b = 1$ has two zeros $t^* = 0.4$ and $t^{**} = 0.75$. Now for $t_0 = 0$, $t \in [0, 1/\gamma[$, we have

$$f^{'''}(t) = \frac{6\gamma^2}{(1-\gamma t)^4} \geq 6\gamma^2 = 1.5 > \|F_x^{'''}\| \text{ for all } x \in D,$$
$$\|F_{x_0}^{\prime -1}F(x_0)\| \leq -\frac{f(t_0)}{f'(t_0)},\ \|F_{x_0}^{\prime -1}\| \leq -\frac{1}{f'(t_0)} \text{ and } \|F_{x_0}^{''}\| \leq f^{''}(t_0).$$

Thus, the hypotheses of Theorem 4.3 are satisfied. Hence, by Theorem 4.3, the sequence $\{x_n\}$ generated by (6) is well defined and remains in $B_{t^*}[x_0]$ and converges to the solution of equation (32).

Finally, we give a numerical example in which the involved operator satisfies all the assumptions of Theorem 4.3 and Theorem 5.1.

Example 6.2 Let $D = X = Y = C([0,1])$ be the space of all continuous real-valued functions defined on $[0,1]$ with norm $\|x\| = \max\{|x(s)| : s \in [0,1]\}$. Consider the following integral equation:

$$x(s) = \frac{\sin(\pi s)}{2} + \frac{1}{24}\int_0^1 \cos(\pi s)\sin(\pi t)x^3(t)\mathrm{d}t. \tag{33}$$

Define $F : D \to Y$ by

$$F(x)(s) = x(s) - \frac{\sin(\pi s)}{2} - \frac{1}{24}\int_0^1 \cos(\pi s)\sin(\pi t)x^3(t)\mathrm{d}t. \tag{34}$$

The first, second and third Fréchet derivative of F at point $x \in D$ are

$$F_x' y(s) = y(s) - \frac{\cos(\pi s)}{8}\int_0^1 \sin(\pi t)x^2(t)y(t)\mathrm{d}t,$$
$$F_x^{''}(yz)(s) = -\frac{\cos(\pi s)}{4}\int_0^1 \sin(\pi t)x(t)y(t)z(t)\mathrm{d}t,$$
$$F_x^{'''}(yzw)(s) = -\frac{\cos(\pi s)}{4}\int_0^1 \sin(\pi t)y(t)z(t)w(t)\mathrm{d}t,$$

for all $y, z, w \in C([0,1])$. Observe that $\|F_x^{''}\|$ is not bounded in $C([0,1])$. For $x_0(s) = 0$ and $y \in X$, we have $F_{x_0}'(y)(s) = (y)(s)$. Thus, we have $F_{x_0}^{\prime -1}U(s) = U(s)$ for all

$U \in C([0,1])$, $\|F_{x_0}'^{-1}\| = 1$, $\|F_{x_0}'^{-1}F(x_0)\| \leq \frac{1}{2} = \eta$, $\|F_{x_0}^{''}\| = 0$, $\|F_{x_0}^{'''}\| \leq \frac{1}{2\pi}$ and

$$\gamma = \max\left\{\frac{\|F_{x_0}^{''}\|}{2}, \left(\frac{\|F_{x_0}^{'''}\|}{6}\right)^{\frac{1}{2}}\right\} = \frac{1}{2\sqrt{3\pi}} = 0.16286750396764.$$

Since $\gamma\eta = 0.08143375198382 \leq 3 - 2\sqrt{2}$, the function $f(t)$ defined by (20) for $b = 1$, it has two zeros $t^* = 0.555192820905552$ and $t^{**} = 2.764787302933913$. Now for $t_0 = 0$, $t \in [0, 1/\gamma[$, we have

$$f^{'''}(t) = \frac{6\gamma^2}{(1-\gamma t)^4} \geq 6\gamma^2 = \frac{1}{2\pi} \geq \|F_x^{'''}\| \text{ for all } x \in D,$$
$$\|F_{x_0}'^{-1}F(x_0)\| \leq -\frac{f(t_0)}{f'(t_0)}, \|F_{x_0}'^{-1}\| \leq -\frac{1}{f'(t_0)} \text{ and } \|F_{x_0}^{''}\| \leq f^{''}(t_0).$$

Thus, the hypotheses of Theorem 4.3 are satisfied. Hence by Theorem 4.3, the sequence $\{x_n\}$ generated by (6) is well defined and remains in $B_{t^*}[x_0]$ and converges to the solution $x^*(s) = \frac{\sin(\pi s)+(128-\sqrt{16383})\cos(\pi s)}{2}$ of equation (33). In the similar way, one can easily observe that all the assumptions of Theorem 5.1 are satisfied that is

$$\|F_{x^*}'^{-1}F_x^{''}\| \leq f^{''}(\|x - x^*\|) \leq f^{''}(\varrho) \text{ for all } x \in B_\varrho(x^*),$$
$$\theta = \kappa t^* = 0.303702708379909 < 1, \text{ here } t^* = r^*.$$

In view of Theorem 5.1, for $\alpha_n = 0.5$, $n \in \mathbb{N}_0$, the following error bounds hold:

$$\|w_{n+1} - x^*\| \leq \kappa\|w_n - x^*\|^2$$

and

$$\|x_{n+1} - x^*\| \leq \kappa\,(1 - 0.5(1-\theta))^2\,\|x_n - x^*\|^2, n \in \mathbb{N}_0,$$

where $\kappa = 0.547024817413695$ and $\kappa\,(1 - 0.5(1-\theta))^2 = 0.232436417985643$.

The following table shows that the sequence generated by (6) is faster than the sequence defined by (2):

S. N.	w_n (Newton method)	x_n (S-iteration process of Newton-like)
0	$0(s)$	$0(s)$
1	$0.5\sin(\pi s)$	$0.5\sin(\pi s) + 0.001953125\cos(\pi s)$
2	$0.5\sin(\pi s) + 0.001953125\cos(\pi s)$	$0.5\sin(\pi s) + 0.001953154803232\cos(\pi s)$
3	$0.5\sin(\pi s) + 0.001953154803232\cos(\pi s)$	$0.5\sin(\pi s) + 0.001953154803232\cos(\pi s)$
4	$0.5\sin(\pi s) + 0.001953154803232\cos(\pi s)$	–

Remark 6.1 (i) In Example 6.1 and 6.2, we cannot apply Newton–Kantorovich Theorem [3, Theorem 6, p.708] because the second derivative of involve operators is not bounded in D and hence there is no guarantee for the convergence of sequence generated by (2) starting from $x_0 = 0$; however, all the conditions of Theorem 4.3 are satisfied. Thus, the sequence $\{x_n\}$ generated by (6) is in $B_{t^*}[x_0]$ and converges to solution $x^* \in B_{t^*}[x_0]$ of equation $F(x) = 0$ staring from initial point $x_0 = 0$.

(ii) In particular, for operator F defined by (34), it is shown in Example 6.2 that our iteration process is faster than the Newton method studied by [22] numerically.

7 Conclusions

We extend and improve the S-iteration process of Newton-like algorithm [27] in the context of Banach spaces and introduced S-iteration process of Newton–Kantorovich like. We have discussed its quadratic convergence analysis using majorizing technique. Numerical results show the effectiveness of our results.

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