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On the convergence of inexact Newton-like methods under mild differentiability conditions

Vipin Kumar Singh

Department of Science and Humanities, Government Polytechnic Korea, Chhattishgarh, 497335, India

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ABSTRACT

In the present paper, we have introduced an inexact Newton-like algorithm and discussed its semilocal convergence analysis under average Lipschitz condition as well as γ-Lipschitz condition for solving generalized operator equations containing non differentiable operators in Banach spaces. Our results extend and improve some well established results in the context of differentiability of involved operators. As special cases of our results, we re-obtain some well established results for the Newton method and inexact Newton method. We apply our result to solve Fredholm integral equations.

1. Introduction

Throughout the paper, we use the following notations: X and Y are Banach spaces, D a nonempty open convex subset of X. In sequel, given any $x \in X$ and $r > 0$, $B_r[x]$ will stand for the set $\{y \in X : \|y - x\| \le r\}$, $B_r(x)$ will stand for the set $\{y \in X : \|y - x\| < r\}$, $B(X, Y)$ will stand for the space of all bounded linear operators from X into Y, F'_x will designate the Fréchet derivative of operator F: $D \to Y$ at point $x \in D$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Let T: $D \to Y$ be an operator. Solving a nonlinear operator equation $T(x) = 0$ in Banach spaces is a basic and very important problem in computational and applied mathematics. A powerful technique for solving such problem is the Newton's method defined by

$$w_{n+1} = w_n - T'^{-1}_{w_n} T(w_n) \text{ for each } n \in \mathbb{N}_0, \tag{1}$$

where $w_0 \in D$ is the initial point. The semilocal convergence analysis of (1) was first studied by Kantorovich and Akilov [1]. Kantorovich's results provide a simple and transparent convergence criterion for the nonlinear operator with bounded second Fréchet derivative T'' or the Lipschitz continuous first Fréchet derivative (see [1,2]). Another important result was introduced by Smale which was presented by Smale in 20th international conference of Mathematician (see [3]), where the concept of an approximate zero of involve operator was proposed and the convergence criteria was provided to evaluate an approximate zero for analytic functions, only depending on the information at initial point. Wang et. al [4–7]. improved and generalized the Smale's result by introducing the notion of γ-condition.

E-mail address: vipinkumarsingh666@mail.com

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In recent years, various results on generalization of Newton's method have been proposed in literature which have either equal or better performance than the Newton's method (see [8–21]). However, there are two disadvantages of the Newton's method. First is to evaluate Fréchet derivative of involved operator T and second is to find exact solution of the following Newton equations:

$$T'_{w_n}(w_{n+1}-w_n) = -T(w_n) \text{ for each } n \in \mathbb{N}_0. \tag{2}$$

To avoid the first disadvantage of Newton's method, a lot of Newton-like methods have been proposed in literature (see [2,13–15]). In many applications when T'_{w_n} is too large and dense, it may not be feasible to solve Eq. (2) exactly. Using linear iterative methods and finding approximate solutions of (2) are logical. To overcome the second disadvantage of Newton's method, linear iterative methods have been extensively developed to find approximate solution of Newton Eq. (2). These methods have been initially introduced by Dembo et al. [22] and are known as Inexact Newton methods. The Inexact Newton method is defined by

$$\begin{cases} x_{n+1} = x_n + v_n, \\ T'_{x_n} v_n = -T(x_n) + y_n \text{ for each } n \in \mathbb{N}_0, \end{cases} \tag{3}$$

where T is a continuously Fréchet differentiable nonlinear operator defined on a nonempty open convex subset D of Banach space X to Banach space Y and $\{y_n\}$ is a sequence of elements of Y which depends on $\{x_n\}$. The convergence behavior of (3) depends on residual controls of $\{y_n\}$. Under the Lipschitz continuity assumption of T', Dembo et al. [22] have discussed the local convergence analysis of inexact Newton method under the conditions that the stopping relative residuals $\{y_n\}$ satisfy $\|y_n\| \le \eta_n \|T(x_n)\|$, where $\{\eta_n\}$ is the sequence of forcing terms. In [23], Ypma used the affine invariant condition $\|T'^{-1}_{x_n} y_n\| \le \eta_n \|T'^{-1}_{x_n} T(x_n)\|$ and analyzed the local convergence analysis of inexact Newton method (3). In [24], Guo used the affine invariant condition $\|T'^{-1}_{x_0} y_n\| \le \eta_n \|T'^{-1}_{x_0} T(x_n)\|$ and analyzed the semilocal convergence analysis of inexact Newton method (3). Avoiding both disadvantages of Newton's method, Morini [25] has proposed the following inexact Newton-like method:

$$\begin{cases} x_{n+1} = x_n + s_n, \\ B_n s_n = -T(x_n) + r_n \text{ for each } n \in \mathbb{N}_0, \end{cases} \tag{4}$$

where $\{r_n\}$ is a sequence in finite Banach space Y while $\{B_n\}$ is a sequence of invertible operator from finite Banach space X to Y and established a local convergence result for inexact Newton like method (4) under the following residual controls:

$$\|P_n r_n\| \le \eta_n \|P_n T(x_n)\| \text{ for each } n \in \mathbb{N}_0, \tag{5}$$

where $\{P_n\}$ is the sequence of invertible operators from Y to X. In [26], Li and Shen generalized the condition (5) by introducing the following condition

$$\|P_n r_n\| \le \eta_n \|P_n T(x_n)\|^{k+1} \text{ for each } n \in \mathbb{N}_0, \tag{6}$$

where $0 \le k \le 1$ and established local convergence result of order $k+1$ for inexact Newton-like method (4). Assuming the Lipschitz continuity condition on $T'^{-1}_{x_0} T'$ and adopting the following residual controls

$$\|T'^{-1}_{x_0} y_n\| \le \eta_n \|T'^{-1}_{x_0} T(x_n)\|^{k+1} \text{ for each } n \in \mathbb{N}_0, \tag{7}$$

Shen and Li [27] established a Kantorovich-type convergence criterion for the inexact Newton method (3). Assuming the residual controls (7) with $k=1$, the γ-condition and the Smale's α-theory have been established by Shen and Li in [28] for the inexact Newton method (3).

Recently, Xu, Xiao and Lio [29] have studied semilocal convergence analysis of the inexact Newton method under the residual controls (7) with $k=1$ and $T'^{-1}_{x_0} T'$ satisfy the weak Lipschitz condition (see [29]).

However, these are rest on the assumption that operator T is Fréchet differentiable. In various cases the nonlinear operator T may not be Fréchet differentiable, though it can be dissolve as a sum of two nonlinear operators such that one is Fréchet differentiable and second is continuous only. It is still possible to use an analogue of inexact Newton method to solve the nonlinear operator equation:

$$F(x) + G(x) = 0, \tag{8}$$

where F: $D \to Y$ is Fréchet differentiable operator at each point of D and G: $D \to Y$ is a continuous operator.

In the present article, motivated by the elegant works in [25–29], we introduce a new Inexact Newton-like method for finding solution of operator Eq. (8).

Algorithm 1.1 (Inexact Newton-like method)**.** Let X and Y be two Banach spaces, D a nonempty open convex subset of X and $x_0 \in D$. Let F: $D \to Y$ be a nonlinear Fréchet differentiable operator and G: $D \to Y$ is a continuous operator and $A(x) \in B(X, Y)$ an approximation of F'_x for $x \in D$ such that $A(x_0)^{-1}$ exists. Let x_0 be an initial guess. For given residual control r_n and the iteration x_n, we defined the next iterate x_{n+1} as follows:

$$\begin{cases} x_{n+1} = x_n + s_n, \\ A(x_n) s_n = -(F(x_n) + G(x_n)) + r_n, \end{cases} \tag{9}$$

where $\{r_n\} \subset Y$ and in general depends on $\{x_n\}$.

The aim of this paper is to study the semilocal convergence analysis of Algorithm 1.1 considering the following residual controls:

$$\|A(x_0)^{-1} r_n\| \le \eta_n \|A(x_0)^{-1}(F(x_n) + G(x_n))\|^2 \text{ for each } n \in \mathbb{N}_0, \tag{10}$$

where $\{\eta_n\}$ is the sequence of forcing terms with $\eta = \sup_{n\in\mathbb{N}_0}\{\eta_n\} < 1$ and $A(x_0)^{-1}F'$ satisfies the following radius Lipschitz condition with L average (see, [30,31]):

$$\|A(x_0)^{-1}(F'_x - F'_y)\| \le \int_{p(x)}^{p(x,y)} L(u)du \text{ for all } x \in B_r(x_0), y \in B_{r-p(x)}[x_0], \tag{11}$$

where $p(x) = \|x - x_0\|$, $p(x, y) = p(x) + \|y - x\| \le r$ and L is positive nondecreasing integrable function on $[0, r]$. We establish the organized convergence criteria which contains Smale type and Katorovich type convergence criteria as special cases. Moreprecisely, when $\eta_n = 0$ for each $n \in \mathbb{N}_0$ and $G(x) = 0$ for each $x \in D$, our Algorithm 1.1 reduces to Newton's method. Our result is more general in nature because it deals with generalized equation (8). It also extend and improves the corresponding results in [28,29,32,33].

The remaining part of the paper is organized as follows: In Section 2, we give some basic definitions and some known results. In Section 3, we discuss the semilocal convergence analysis of Algorithm 1.1 under Lipschitz condition (11), we also show that our result reduces to some well established results. In Section 4, we study the semilocal convergence analysis of our Algorithm 1.1 under the γ-Lipschitz condition. We also discussed some special cases of our result. Finally, we conclude our work with an example, where theorems [32, Theorem 2.9] and [28, Theorem 3.1] fails and our result holds good.

2. Preliminaries

In this section, we give some known results which are useful in the remaining part of the paper.

Lemma 2.1 [29]**.** *Let λ, b, r, σ_0, and c be positive constants with $c \ge 1$, $0 < b \le 1$, and $\sigma_0 \ge 0$. Let L be a positive nondecreasing integrable function defied on $[0, r]$ and $\varphi : [0, r] \to \mathbb{R}$ be a function defined by*

$$\varphi(t) = \lambda - bt + \sigma_0 t^2 + c\int_0^t L(u)(t-u)du. \tag{12}$$

Assume that r^* and b^* are real numbers given by:

$$r^* = \sup\left\{t \in (0, r) : c\int_0^t L(u)du + 2\sigma_0 t \le b\right\}, \tag{13}$$

$$b^* = br^* - \sigma_0 {r^*}^2 - c\int_0^{r^*} L(u)(r^* - u)du. \tag{14}$$

If $\lambda \le b^*$, then φ is strictly decreasing on $[0, r^*]$ and has exact one zero t^* on $[0, r^*]$ with $\lambda < t^*$.

Lemma 2.2 [29]**.** *Assume that all the assumption of Lemma 2.1 hold and L_0 is a positive nondecreasing integrable function defined on $[0, r]$. Define $\psi : [0, r] \to \mathbb{R}$ by*

$$\psi(t) = \lambda - t + c\int_0^t L_0(u)(t-u)du. \tag{15}$$

Suppose $L_0(t) \le L(t)$ for each $t \in [0, r]$ and t^* is the positive solution of $\varphi(t) = 0$ on $[0, r^*]$. Then the sequence $\{t_n\}$ generated by

$$\begin{cases} t_0 = 0, \\ t_{n+1} = t_n - \frac{\varphi(t_n)}{\psi'(t_n)}, n \in \mathbb{N} \end{cases} \tag{16}$$

is strictly increasing and converges to t^*. Moreover,

$$t_{n+1} < t_n < t^*, n \in \mathbb{N}_0.$$

Lemma 2.3 (Rall [34], p. 50)**.** *Let L be a bounded linear operator on a Banach space X. Then, L^{-1} exists if and only if there is a bounded linear operator M in X such that M^{-1} exists, and*

$$\|M - L\| < \frac{1}{\|M^{-1}\|}.$$

If L^{-1} exists, then

$$\left\|L^{-1}\right\| \le \frac{\left\|M^{-1}\right\|}{1 - \|1 - M^{-1}L\|} \le \frac{\left\|M^{-1}\right\|}{1 - \|M^{-1}\|\|M - L\|}.$$

3. Semilocal convergence analysis of Algorithm 1.1 under Lipschitz condition (11)

Lemma 3.1. *Let X and Y be two Banach spaces, D a nonempty open subset of X and $A(x) \in B(X, Y)$ such that $A(x_0)^{-1}$ exists for some $x_0 \in D$. Let $r \geq 0$ be such that $B_r(x_0) \subset D$ and L_0 a positive nondecreasing integrable function defined on $[0, r]$. Assume that the following conditions hold:*

(C1) $\|A(x_0)^{-1}(A(x) - A(x_0))\| \leq \int_0^{\|x-x_0\|} L_0(u)du$ *for each* $x \in B_r(x_0)$;
(C2) $\int_0^r L_0(u)du < 1$.

Then, for each $x \in B_r(x_0)$, $A(x)$ is invertible and

$$\|A(x)^{-1}A(x_0)\| \leq \frac{1}{1 - \int_0^{\|x-x_0\|} L_0(u)du} \leq -\frac{1}{\psi'(\|x - x_0\|)}, \tag{17}$$

where ψ is defined by (15).

Proof. For $x \in B_r(x_0)$, we have

$$\begin{aligned}\|I - A(x_0)^{-1}A(x)\| &= \|A(x_0)^{-1}(A(x) - A(x_0))\| \\ &\leq \int_0^{\|x-x_0\|} L_0(u)du < 1.\end{aligned}$$

Therefore, by Lemma 2.3, we have

$$\|A(x)^{-1}A(x_0)\| \leq \frac{1}{1 - \int_0^{\|x-x_0\|} L_0(u)du}. \tag{18}$$

For $x \in B_r(x_0)$, we have

$$-\psi'(\|x - x_0\|) = 1 - c\int_0^{\|x-x_0\|} L_0(u)du \leq 1 - \int_0^{\|x-x_0\|} L_0(u)du.$$

This implies that

$$\frac{1}{1 - \int_0^{\|x-x_0\|} L_0(u)du} \leq -\frac{1}{\psi'(\|x - x_0\|)}. \tag{19}$$

Thus (17) follows from (18) and (19). This completes the proof. □

Proposition 3.2. *Let X and Y be two Banach spaces, D a nonempty open convex subset of X and $x_0 \in D$. Let $F: D \to Y$ be a nonlinear Fréchet differentiable operator and $G: D \to Y$ a continuous operator and $A(x) \in B(X, Y)$ an approximation of F'_x for $x \in D$ such that $A(x_0)^{-1}$ exists. Assume that F'_x satisfies the Lipschitz condition (11). We also assume that (C1), (C2) and the following conditions hold:*

(C3) $\|A(x_0)^{-1}(G(x) - G(y))\| \leq K\|x - y\|$ *for each* $x, y \in D$ *and for some* $K \geq 0$;
(C4) $\|A(x_0)^{-1}(A(x) - F'_x)\| < \delta$ *for each* $x \in D$ *and for some* $\delta \geq 0$.

Let $r > 0$ be such that $B_r(x_0) \subseteq D$ and $R: B_r(x_0) \to Y$ be an operator satisfies the following condition:

$$\|A(x_0)^{-1}R(x)\| \leq \eta_x\|A(x_0)^{-1}(F(x) + G(x))\|^2 \text{ for each } x \in B_r(x_0),$$

where $0 \leq \eta_x < 1$ such that $\sqrt{\eta_x}\|A(x_0)^{-1}(F(x) + G(x))\| \leq 1$ for each $x \in B_r(x_0)$.
Define $T: B_r(x_0) \to X$ and $S: B_r(x_0) \to X$ by

$$\begin{cases} T(x) = x + S(x), \\ A(x)(S(x)) = -[F(x) + G(x)] + R(x), \end{cases} \tag{20}$$

for each $x \in B_r(x_0)$. Then for each $x \in B_r(x_0)$ with $T(x) \in B_r(x_0)$, we have

(a) $\|A(x_0)^{-1}[F(T(x)) + G(T(x))]\| \leq \int_0^1 \int_{\|x-x_0\|}^{\|x-x_0\|+t\|T(x)-x\|} L(u)\|T(x) - x\|dudt + (K + \delta)\|T(x) - x\| + \eta_x\frac{\left(1+\int_0^{\|x-x_0\|} L_0(u)du\right)^2}{(1-\sqrt{\eta_x})^2}\|T(x) - x\|^2$
(b) $\|T(x) - x\| \leq (1 + \sqrt{\eta_x})\|A(x_0)^{-1}(F(x) + G(x))\|\|A(x)^{-1}A(x_0)\|$.

Proof. Let $x \in D$ and using (20), we have

$$\begin{aligned}&\|A(x_0)^{-1}[F(T(x)) + G(T(x))]\| \\ &= \|A(x_0)^{-1}[F(T(x)) + G(T(x)) - F(x) - G(x) - A(x)(T(x) - x)] + A(x_0)^{-1}R(x)\|\end{aligned}$$

$$
\begin{aligned}
&\le \int_0^1 \left(\|A(x_0)^{-1}(F'_{x+t(T(x)-x)} - F'_x)\| + \|A(x_0)^{-1}(F'_x - A(x))\|\right)\|T(x)-x\|dt \\
&\quad + \|A(x_0)^{-1}(G(T(x)) - G(x))\| + \|A(x_0)^{-1}R(x)\| \\
&\le \int_0^1 \int_{\|x-x_0\|}^{\|x-x_0\|+t\|T(x)-x\|} L(u)\|T(x)-x\|dudt + (K+\delta)\|T(x)-x\| + \|A(x_0)^{-1}R(x)\| \\
&\le \int_0^1 \int_{\|x-x_0\|}^{\|x-x_0\|+t\|T(x)-x\|} L(u)\|T(x)-x\|dudt \\
&\quad + (K+\delta)\|T(x)-x\| + \eta_x\|A(x_0)^{-1}(F(x)+G(x))\|^2. \qquad (21)
\end{aligned}
$$

Again from (20), we have

$$
\begin{aligned}
\|A(x_0)^{-1}A(x)(T(x)-x)\| &\ge \|A(x_0)^{-1}(F(x)+G(x))\| - \|A(x_0)^{-1}R(x)\| \\
&\ge \|A(x_0)^{-1}(F(x)+G(x))\| - \sqrt{\eta_x}\|A(x_0)^{-1}(F(x)+G(x))\| \\
&= (1-\sqrt{\eta_x})\|A(x_0)^{-1}(F(x)+G(x))\|,
\end{aligned}
$$

which implies that

$$
\|A(x_0)^{-1}(F(x)+G(x))\| \le \frac{\|A(x_0)^{-1}A(x)\|\|T(x)-x\|}{1-\sqrt{\eta_x}}. \qquad (22)
$$

Now using condition (C1), we have

$$
\|A(x_0)^{-1}A(x)\| = \|I + A(x_0)^{-1}(A(x)-A(x_0))\| \le 1 + \int_0^{\|x-x_0\|} L_0(u)du. \qquad (23)
$$

Thus part (a) follows from (21), (22) and (23).

(b) By Lemma 3.1, we have $A(x)$ is invertible for each $x \in D$. Using (20), we have

$$
\begin{aligned}
\|T(x)-x\| &\le \|A(x)^{-1}A(x_0)\|\left(\|A(x_0)^{-1}(F(x)+G(x))\| + \|A(x_0)^{-1}R(x)\|\right) \\
&\le \|A(x)^{-1}A(x_0)\|\left(\|A(x_0)^{-1}(F(x)+G(x))\| + \sqrt{\eta_x}\|A(x_0)(F(x)+G(x))\|\right) \\
&= (1+\sqrt{\eta_x})\|A(x_0)^{-1}(F(x)+G(x))\|\|A(x)^{-1}A(x_0)\|.
\end{aligned}
$$

This completes the proof. □

Lemma 3.3. *Let X and Y be two Banach spaces, D a nonempty open convex subset of X and $x_0 \in D$. Let F: $D \to Y$ be a nonlinear Fréchet differentiable operator and G: $D \to Y$ a continuous operator and $A(x) \in B(X, Y)$ an approximation of F'_x for $x \in D$ such that $A(x_0)^{-1}$ exists. Suppose that the condition (10) and all the assumptions of Lemmas 2.2 and 3.1 hold. Assume that F'_x satisfies the Lipschitz condition (11). We also assume that (C3), (C4) and the following conditions hold:*

(C5) $\|A(x_0)^{-1}(F(x_0)+G(x_0))\| = \beta$ *for some* $\beta > 0$;
(C6) $\sqrt{\eta}\|A(x_0)^{-1}(F(x_0)+G(x_0))\| \le 1$;
(C7) $c = 1+\sqrt{\eta}, \lambda = c\beta, \kappa = K+\delta, b = 1-(1+\sqrt{\eta})\kappa$ *and* $\sigma_0 = \frac{\eta(1+\sqrt{\eta})\left(1+\int_0^r L_0(u)du\right)^2}{(1-\sqrt{\eta})^2}$.

Then the sequence $\{x_n\}$ generated by Algorithm 1.1 is well defined. Moreover, the following estimates hold:

(a) $\|A(x_n)^{-1}A(x_0)\| \le -\psi'(t_n)^{-1}$,
(b) $(1+\sqrt{\eta})\|A(x_0)^{-1}(F(x_n)+G(x_n))\| \le \varphi(t_n)$,
(c) $\sqrt{\eta}\|A(x_0)^{-1}(F(x_n)+G(x_n))\| \le 1$,
(d) $\|x_{n+1}-x_n\| \le (t_{n+1}-t_n)\frac{\|x_n-x_{n-1}\|}{t_n-t_{n-1}}$,
(e) $\frac{\|A(x_0)^{-1}(F(x_n)+G(x_n))\|}{\|A(x_0)^{-1}(F(x_{n-1})+G(x_{n-1}))\|} \le \frac{\varphi(t_n)}{\varphi(t_{n-1})}$ *for all* $n \in \mathbb{N}_0$,

where $\{t_n\}$ is a real sequence defined by (16).

Proof Set $s_n = S(x_n)$, $r_n = R(x_n)$ and $\eta_{x_n} = \eta_n$ in Proposition 3.2, then the sequence $\{x_n\}$ generated by Algorithm 1.1 is same as $x_{n+1} = T(x_n), n \in \mathbb{N}_0$. The proof of this Lemma will be complete into the following steps:

Step I. Estimates $(a)-(e)$ hold for $n = 1$.

For $x_0 \in D$, using Proposition 3.2(b), we have

$$
\begin{aligned}
\|x_1 - x_0\| &= \|T(x_0) - x_0\| \\
&= (1+\sqrt{\eta})\|A(x_0)^{-1}(F(x_0)+G(x_0))\| \\
&\le (1+\sqrt{\eta})\beta = t_1.
\end{aligned}
$$

From (17), we have $\|A(x_1)^{-1}A(x_0)\| \le -\psi'(\|x_1-x_0\|)^{-1} \le -\psi'(t_1)^{-1}$.

Using Proposition 3.2 (a), we have

$$\begin{aligned}\|A(x_0)^{-1}(F(x_1)+G(x_1))\| &= \|A(x_0)^{-1}[F(T(x_0))+G(T(x_0))]\| \\ &\le \int_0^1 \int_0^{t\|T(x_0)-x_0\|} L(u)dudt\|T(x_0)-x_0\| \\ &\quad +\frac{\eta}{(1-\sqrt{\eta})^2}\|T(x_0)-x_0\|^2+\kappa\|T(x_0)-x_0\| \end{aligned} \tag{24}$$

and

$$\begin{aligned}&(1+\sqrt{\eta})\|A(x_0)^{-1}(F(x_1)+G(x_1))\| \\ &\le (1+\sqrt{\eta})\int_0^1\int_0^{t\|x_1-x_0\|} L(u)du\|x_1-x_0\|dt+\frac{(1+\sqrt{\eta})\eta}{(1-\sqrt{\eta})^2}\|x_1-x_0\|^2+(1+\sqrt{\eta})\kappa\|x_1-x_0\| \\ &\le \left(c\int_0^{t_1}(t_1-v)L(v)dv+\sigma_0(t_1-t_0)^2+(1+\sqrt{\eta})\kappa(t_1-t_0)\right)\frac{\|x_1-x_0\|}{t_1-t_0} \\ &= \varphi(t_1)\frac{\|x_1-x_0\|}{t_1-t_0}. \end{aligned} \tag{25}$$

Now

$$\sqrt{\eta}\|A(x_0)^{-1}(F(x_1)+G(x_1))\| \le \frac{\lambda\sqrt{\eta}}{1+\sqrt{\eta}} = \sqrt{\eta}\|A(x_0)^{-1}(F(x_0)+G(x_0))\| \le 1.$$

From (16), (25) and Proposition 3.2(b), we have

$$\begin{aligned}\frac{(1+\sqrt{\eta})\|A(x_0)^{-1}(F(x_1)+G(x_1))\|}{(1+\sqrt{\eta})\|A(x_0)^{-1}(F(x_0)+G(x_0))\|} &\le \frac{\varphi(t_1)(1+\sqrt{\eta})\|A(x_0)^{-1}(F(x_0)+G(x_0))\|}{(1+\sqrt{\eta})\|A(x_0)^{-1}(F(x_0)+G(x_0))\|(t_1-t_0)} \\ &= -\frac{\varphi(t_1)}{\psi'(t_0)(t_1-t_0)} \\ &= \frac{\varphi(t_1)}{\varphi(t_0)}. \end{aligned}$$

Using (16), (17), (25) and Proposition 3.2, we have

$$\begin{aligned}\|x_2-x_1\| &\le (1+\sqrt{\eta})\|A(x_1)^{-1}A(x_0)\|\|A(x_0)^{-1}(F(x_1)+G(x_1))\| \\ &\le -\frac{\varphi(t_1)}{\psi'(t_1)}\frac{\|x_1-x_0\|}{t_1-t_0} \\ &= (t_2-t_1)\frac{\|x_1-x_0\|}{t_1-t_0} \le t_2-t_1. \end{aligned}$$

Step II. Estimates $(a)-(e)$ hold for $n\in\mathbb{N}$.
Let $k\ge 1$ be a positive integer. Assume that the following hold for $n=1,2,\ldots,k$

$$\sqrt{\eta}\|A(x_0)^{-1}(F(x_{k-1})+G(x_{k-1}))\| \le 1 \text{ and } \|x_k-x_{k-1}\| \le t_k-t_{k-1}. \tag{26}$$

From (17) and (26), we have

$$\|A(x_k)^{-1}A(x_0)\| \le -\psi'(t_k)^{-1}.$$

From Proposition 3.2 (a), we have

$$\begin{aligned}\|A(x_0)^{-1}(F(x_k)+G(x_k))\| &= \|A(x_0)^{-1}(F(T(x_{k-1}))+G(T(x_{k-1})))\| \\ &\le \int_0^1\int_{\|x_{k-1}-x_0\|}^{\|x_{k-1}-x_0\|+t\|x_k-x_{k-1}\|} L(u)\|x_k-x_{k-1}\|dudt+\kappa\|x_k-x_{k-1}\| \\ &\quad +\eta\frac{\left(1+\int_0^{\|x_{k-1}-x_0\|}L_0(u)du\right)^2}{(1-\sqrt{\eta})^2}\|x_k-x_{k-1}\|^2 \end{aligned} \tag{27}$$

and

$$\begin{aligned}(1+\sqrt{\eta})\|A(x_0)^{-1}(F(x_k)+G(x_k))\| \le &\Bigg((1+\sqrt{\eta})\int_0^1\int_{t_{k-1}}^{t_{k-1}+t(t_k-t_{k-1})} L(u)du(t_k-t_{k-1})dt \\ &+\sigma_0(t_k-t_{k-1})^2+(1+\sqrt{\eta})\kappa(t_k-t_{k-1})\Bigg)\frac{\|x_k-x_{k-1}\|}{t_k-t_{k-1}} \end{aligned}$$

$$
\begin{aligned}
&= \left(c\int_0^{t_k-t_{k-1}} L(t_{k-1}+v)(t_k-t_{k-1}-v)dv + \sigma_0(t_k-t_{k-1})^2 \right.\\
&\quad \left. +(1+\sqrt{\eta})\kappa(t_k-t_{k-1}) \right) \frac{\|x_k-x_{k-1}\|}{t_k-t_{k-1}}\\
&= \left(c\int_{t_{k-1}}^{t_k} L(v)(t_k-v)dv + \sigma_0(t_k-t_{k-1})^2 + (1+\sqrt{\eta})\kappa(t_k-t_{k-1}) \right) \frac{\|x_k-x_{k-1}\|}{t_k-t_{k-1}}\\
&\leq (\phi(t_k)-\phi(t_{k-1})-\psi'(t_{k-1})(t_k-t_{k-1})) \frac{\|x_k-x_{k-1}\|}{t_k-t_{k-1}}\\
&= \varphi(t_k) \frac{\|x_k-x_{k-1}\|}{t_k-t_{k-1}}. \qquad (28)
\end{aligned}
$$

Since φ is decreasing on $[0, r^*]$ then

$$(1+\sqrt{\eta})\|A(x_0)^{-1}[F(x_k)+G(x_k)]\| \leq \varphi(t_k) \leq \varphi(t_0) = \lambda$$

and

$$
\begin{aligned}
\sqrt{\eta}\|A(x_0)^{-1}[F(x_k)+G(x_k)]\| &\leq \frac{\lambda\sqrt{\eta}}{1+\sqrt{\eta}}\\
&= \sqrt{\eta}\beta = \sqrt{\eta}\|A(x_0)^{-1}(F(x_0)+G(x_0))\| < 1.
\end{aligned}
$$

From Proposition 3.2 (b), we have

$$\|x_k-x_{k-1}\| \leq (1+\sqrt{\eta})\|A(x_0)^{-1}(F(x_{k-1})+G(x_{k-1}))\|\|A(x_{k-1})^{-1}A(x_0)\|. \qquad (29)$$

From (16), (28) and (29), we have

$$
\begin{aligned}
\frac{(1+\sqrt{\eta})\|A(x_0)^{-1}(F(x_k)+G(x_k))\|}{(1+\sqrt{\eta})\|A(x_0)^{-1}(F(x_{k-1})+G(x_{k-1}))\|} &\leq \frac{\varphi(t_k)(1+\sqrt{\eta})\|A(x_0)^{-1}(F(x_{k-1})+G(x_{k-1}))\|}{(1+\sqrt{\eta})\|A(x_0)^{-1}(F(x_{k-1})+G(x_{k-1}))\|(t_k-t_{k-1})}\\
&\leq -\frac{\varphi(t_k)}{\psi'(t_{k-1})(t_k-t_{k-1})}\\
&= \frac{\varphi(t_k)}{\varphi(t_{k-1})}.
\end{aligned}
$$

From (16), (29) and Proposition 3.2, we have

$$
\begin{aligned}
\|x_{k+1}-x_k\| &\leq (1+\sqrt{\eta})\|A(x_k)^{-1}A(x_0)\|\|A(x_0)^{-1}(F(x_k)+G(x_k))\|\\
&\leq -\frac{\varphi(t_k)}{\psi'(t_k)}\frac{\|x_k-x_{k-1}\|}{t_k-t_{k-1}}\\
&= \frac{t_{k+1}-t_k}{t_k-t_{k-1}}\|x_k-x_{k-1}\| \leq t_{k+1}-t_k.
\end{aligned}
$$

Thus, (26) holds for $n = k+1$. Therefore, by mathematical induction (26) is true for each $n \in \mathbb{N}_0$. Hence (a)-(e) are hold for each $n \in \mathbb{N}_0$. This completes the proof. □

Theorem 3.4. *Assume that all the assumption of Lemma 3.3 hold. Suppose that $\lambda \leq \min\{\frac{1}{\sqrt{\eta}}, b^*\}$ and $B_{t^*}[x_0] \subseteq B_r(x_0)$. Then the sequence $\{x_n\}$ generated by Algorithm 1.1 is well defined, remains in $B_{t^*}[x_0]$ and converges to the solution $x^* \in B_{t^*}[x_0]$ of (8). Moreover,*

$$\|x_{n+1}-x_n\| \leq t_{n+1}-t_n \text{ and } \|x^*-x_n\| \leq t^*-t_n, n \in \mathbb{N}_0.$$

Proof. Note that $\lambda \leq \min\{\frac{1}{\sqrt{\eta}}, b^*\}$. This implies that Lemmas 2.1 and 2.2 is applicable. It is clear from Lemma 3.3 that the sequence $\{x_n\}$ is well defined and the following assertions hold:

$$\sqrt{\eta}\|A(x_0)^{-1}(F(x_n)+G(x_n))\| \leq 1 \text{ and}$$

$$\|x_{n+1}-x_n\| \leq t_{n+1}-t_n, n \in \mathbb{N}_0.$$

Consequently, for n, $m \geq 0$, we have

$$\|x_{n+m}-x_n\| \leq \sum_{k=n}^{m+n-1}\|x_{k+1}-x_k\| \leq t_{n+m}-t_n. \qquad (30)$$

Note that the sequence $\{t_n\}$ is convergent and hence sequence $\{x_n\}$ is a Cauchy sequence so it is converges to some point x^*.

Taking limit as $m\to\infty$ in (30), we have

$$\|x^* - x_n\| \le t^* - t_n. \tag{31}$$

Taking $n = 0$ in (30) and (31), we obtain

$$\|x_m - x_0\| \le t_m \le t^* \text{ and } \|x^* - x_0\| \le t^*.$$

This completes the proof. □

Remark 3.5.

(1) Theorem 3.4 is more general in nature because it deals with generalized equation (8), where involved operator is not necessarily Fréchet differentiable.
(2) Theorems [32, Theorem 2.9] and [28, Theorem 3.1] are not applicable for operator equation (8), but our Theorem 3.4 works well. Supporting example is given in Section 4.

If we take $A(x) = F'_x$ for each $x\in D$ in Theorem 3.4, we have the following result.

Corollary 3.6. *Let X and Y be two Banach spaces, D a nonempty open convex subset of X and $x_0\in D$. Let F: $D\to Y$ be a nonlinear Fréchet differentiable operator such that $F'^{-1}_{x_0}$ exists and G: $D\to Y$ a continuous operator. Assume that all the assumption of Lemma 2.2 hold and $\lambda \le \min\{\frac{1}{\sqrt{\eta}}, b^*\}$. We also assume that the conditions (10), (11), $(C1)$, $(C3)$, $(C5)$, $(C6)$ hold for $A(x) = F'_x$ on $B_{t^*}[x_0] \subseteq B_r(x_0)$ and $(C7)$ holds with $\delta = 0$. Then sequence $\{x_n\}$ generated by Algorithm 1.1 is well defined, remains in $B_{t^*}[x_0]$ and converges to the solution $x^* \in B_{t^*}[x_0]$ of (8). Moreover,*

$$\|x_{n+1} - x_n\| \le t_{n+1} - t_n \text{ and } \|x^* - x_n\| \le t^* - t_n, n \in \mathbb{N}_0.$$

Proof. The proof of Corollary 3.6 follows from Theorem 3.4. □

In the special case when $A(x) = F'_x, G(x) = 0$ for each $x\in D$ and $L(t) = L_0(t)$ for each $t\in[0, r]$, then $\kappa = 0$ and Corollary 3.6 reduces to the main result [29, Theorem 3.1].

Corollary 3.7 [29, Theorem 3.1]**.** *Let X and Y be two Banach spaces, D a nonempty open convex subset of X and $x_0\in D$. Let F: $D\to Y$ be a nonlinear Fréchet differentiable operator such that $F'^{-1}_{x_0}$ exists. Assume that all the assumption of Lemma 2.2 hold and $\lambda \le \min\{\frac{1}{\sqrt{\eta}}, b^*\}$. We also assume that the conditions (10), (11), $(C1)$, $(C5)$ and $(C6)$ hold for $A(x) = F'_x$ and $G(x) = 0$ for each $x\in D$. Let $c = 1 + \sqrt{\eta}, \lambda = c\beta,\ \sigma_0 = \frac{\eta(1+\sqrt{\eta})\left(1+\int_0^r L_0(u)du\right)^2}{(1-\sqrt{\eta})^2}$ and $B_{t^*}[x_0] \subseteq B_r(x_0)$. Then the sequence $\{x_n\}$ generated by Algorithm 1.1 is well defined, remains in $B_{t^*}[x_0]$ and converges to the solution $x^* \in B_{t^*}[x_0]$ of $F(x) = 0$. Moreover,*

$$\|x_{n+1} - x_n\| \le t_{n+1} - t_n \text{ and } \|x^* - x_n\| \le t^* - t_n, n \in \mathbb{N}_0.$$

Set $A(x) = F'_x$ for each $x\in D$, $\eta_n = 0,\ n \in \mathbb{N}_0$ and $L(t) = L_0(t) = K'$ for each $0\le t\le r$ in Theorem 3.4, we have the following result:

Corollary 3.8. *Let X and Y be two Banach spaces, D a nonempty open convex subset of X and $x_0\in D$. Let F: $D\to Y$ be Fréchet differentiable operator such that $F'^{-1}_{x_0}$ exists and G: $D\to Y$ a continuous operator. Assume that F and G satisfies the following conditions:*

(C8) $\|F'^{-1}_{x_0}(F(x_0) + G(x_0))\| = \beta$ *for some* $\beta > 0$;
(C9) $\|F'^{-1}_{x_0}(G(x) - G(y))\| \le K\|x - y\|$ *for each* $x, y\in D$ *and for some* $K\ge 0$;
(C10) $\|F'^{-1}_{x_0}(F'_x - F'_y)\| \le K'\|x - y\|$ *for each* $x, y\in D$ *and for some* $K' \ge 0$.
(C11) $h = \frac{\beta K'}{(1-K)^2} \le \frac{1}{2}$.

Let $B_{s^*}[x_0] \subset D$, where $s^* = \frac{1-\sqrt{1-2h}}{h}\frac{\beta}{1-K}$. Then, the sequence $\{x_n\}$ generated by Newton-like method

$$x_{n+1} = x_n - F'^{-1}_{x_n}(F(x_n) + G(x_n)), n \in \mathbb{N}_0$$

is well defined, remains in $B_{s^*}[x_0]$ and converges to the solution of equation (8).

Proof. Set $L(t) = L_0(t) = K'$ for each $0\le t\le r$, $\eta_n = 0, n \in \mathbb{N}_0$ and $A(x) = F'_x$ for each $x\in D$ in Theorem 3.4, we obtain $\lambda = \beta, \eta = 0, b = 1 - K, \sigma_0 = 0$ and $c = 1$. Thus Eqs. (12) and (15) reduces to

$$\varphi(t) = \frac{K't^2}{2} - (1 - K)t + \beta \text{ and } \psi(t) = \frac{K't^2}{2} - t + \beta$$

respectively. One can easily observe that equation $\varphi'(t) = 0$ has solution $r^* = \frac{1-K}{K'}$. Condition $(C11)$ implies that $\beta = \lambda \le b^*$. Thus the equation $\varphi(t) = 0$ has a solution $s^* = \frac{1-\sqrt{1-2h}}{h}\frac{\beta}{1-K}$ such that $s^* \le r^*$. Since all the assumption of Theorem 3.4 are satisfied. Therefore, proof of Corollary 3.8 follows from Theorem 3.4. This completes the proof. □

4. Semilocal convergence analysis of Algorithm 1.1 under γ- Lipschitz condition

Let γ, b and c be positive real numbers with $0<b\leq 1$ the function $\varphi_\gamma : [0, \frac{1}{\gamma}) \to \mathbb{R}$ defined by

$$\varphi_\gamma(t) = \lambda - bt + \frac{c\gamma t^2}{1-\gamma t} \tag{32}$$

was initially introduced by Wang [4,5] for $c=b=1$. The function φ_γ was also used by Argyros et al. [32] and Shen and Li [27,28] for $b=1$ as a majorizing function for the study of convergence analysis of Algorithm 1.1.

Let γ and γ_0 be two positive real numbers such that $\gamma_0 \leq \gamma$. Define $L : [0, 1/\gamma) \to \mathbb{R}$ and $L_0 : [0, 1/\gamma_0) \to \mathbb{R}$ by

$$L(t) = \frac{2\gamma}{(1-\gamma t)^3}, \tag{33}$$

$$L_0(t) = \frac{2\gamma_0}{(1-\gamma_0 t)^3}, \tag{34}$$

Clearly $L_0(t)\leq L(t)$ for each $t\in[0, 1/\gamma)$. Taking above $L_0(t)$, $L(t)$ and $\sigma_0 = 0$ in Lemma 2.2, we have the following result:

Lemma 4.1. *Let γ, b and c be positive real numbers with $0<b\leq 1$. Let φ_γ be defined by (32) and define $\psi_{\gamma_0} : [0, \frac{1}{\gamma_0}) \to \mathbb{R}$ by*

$$\psi_{\gamma_0}(t) = \lambda - t + \frac{c\gamma_0 t^2}{1-\gamma_0 t}. \tag{35}$$

If

$$\lambda\gamma \leq b + 2c - 2\sqrt{c(c+b)}. \tag{36}$$

Then, we obtain the following:

(a) $\varphi'_\gamma(t) = 0$ *has a unique solution* $r^* = (1-\sqrt{\frac{c}{c+b}})\frac{1}{\gamma}$ *in* $(0, \frac{1}{\gamma})$.
(b) *Equation* $\varphi_\gamma(t) = 0$ *has two real roots* $t^* = \frac{\lambda\gamma+b-\sqrt{(\lambda\gamma+b)^2-4\lambda(b+c)\gamma}}{2(b+c)\gamma}$ and $t^{**} = \frac{\lambda\gamma+b+\sqrt{(\lambda\gamma+b)^2-4\lambda(b+c)\gamma}}{2(b+c)\gamma}$ *satisfies* $\lambda<t^*\leq r^*\leq t^{**}$.
(c) *The sequence* $\{t_n\}$ *generated by*

$$\begin{cases} t_0 = 0, \\ t_{n+1} = t_n - \frac{\varphi_\gamma(t_n)}{\psi'_{\gamma_0}(t_n)}, n\in\mathbb{N} \end{cases} \tag{37}$$

is nondecreasing and converges to t^*.

Lemma 4.2. *Let t^* be a solution of $\varphi_\gamma(t) = 0$ and $\{t_n\}$ be a real sequence generated by (37). Suppose that (36) holds. Then following assertions hold:*

$$\frac{t^*-t_n}{t^*-t_{n-1}} \leq \theta^{2^{n-1}}, \quad \frac{t_{n+1}-t_n}{t_n-t_{n-1}} \leq \theta^{2^{n-1}} \text{ and } \frac{\varphi_\gamma(t_n)}{\varphi_\gamma(t_{n-1})} \leq \theta^{2^{n-1}}, \tag{38}$$

where

$$\theta = \frac{b-\lambda\gamma-\sqrt{(b+\lambda\gamma)^2-4(b+c)\lambda\gamma}}{b-\lambda\gamma+\sqrt{(b+\lambda\gamma)^2-4(b+c)\lambda\gamma}}.$$

Proof. The proof of this lemma follows from [28, Lemma 2.3]. □

Definition 4.3. Let X and Y be two Banach spaces, D a nonempty open convex subset of X and $x_0\in D$. Let F: $D\to Y$ be a nonlinear Fréchet differentiable operator and $A(x)\in B(X, Y)$ an approximation of F'_x for $x\in D$ such that $A(x_0)^{-1}$ exists. Let $\gamma>0$ and let $0<r\leq\frac{1}{\gamma}$ be such that $B_r(x_0)\subseteq D$. The operator F is said to satisfy the γ-Lipschitz condition if

$$\|A(x_0)^{-1}(F'_x - F'_y)\| \leq \int_{p(x)}^{p(x,y)} L(u)du \text{ for each } x\in B_r(x_0) \text{ and } y\in B_{r-p(x)}[x_0], \tag{39}$$

where $p(x) = \|x-x_0\|$, $p(x, y) = p(x) + \|y-x\| \leq r$ and L is defined by (33).

Lemma 4.4. *Let γ_0 be a positive real number and $r\leq(1-\frac{c}{\sqrt{c+b}})\frac{1}{\gamma_0}$. Let X and Y be two Banach spaces, D a nonempty open subset of X and $A(x)\in B(X, Y)$ an approximation of F'_x for $x\in D$ such that $A(x_0)^{-1}$ exists for some $x_0\in D$. Assume that L_0 is defined by (34) and condition (C1) holds. Then, for each $x\in B_r(x_0)$, we have*

$$\|A(x)^{-1}A(x_0)\| \leq \left(2 - \frac{1}{(1-\gamma_0\|x-x_0\|)^2}\right)^{-1} \leq -\frac{1}{\psi'_{\gamma_0}(\|x-x_0\|)},$$

where ψ_{γ_0} is defined by (35).

Lemma 4.5. *Let X and Y be two Banach spaces, D a nonempty open convex subset of X and $x_0 \in D$. Let F: $D \to Y$ be a nonlinear Fréchet differentiable operator and G: $D \to Y$ a continuous operator and $A(x) \in B(X, Y)$ an approximation of F'_x for $x \in D$ such that $A(x_0)^{-1}$ exists. Suppose that the condition (10) and all the assumptions of Lemmas 4.1 and 4.4 hold. Assume that F′ satisfies the γ-Lipschitz condition (4.3), we also assume that the conditions (C3) – (C6) hold on $B_r(x_0) \subseteq D, 0 < r \leq \frac{1}{\gamma}$. Let $\lambda = (1 + \sqrt{\eta})\beta, \kappa = K + \delta, b = 1 - (1 + \sqrt{\eta})\kappa$ and $c = 1 + \sqrt{\eta} + \frac{\sqrt{2}\eta(1+\sqrt{\eta})}{\gamma(1-\sqrt{\eta})^2}$. Then the sequence $\{x_n\}$ generated by Algorithm 1.1 is well defined. Moreover, the following assertions hold:*

(a) $\|A(x_n)^{-1}A(x_0)\| \leq -\psi'_{\gamma_0}(t_n)^{-1}$,
(b) $(1 + \sqrt{\eta})\|A(x_0)^{-1}(F(x_n) + G(x_n))\| \leq \varphi_\gamma(t_n)$,
(c) $\sqrt{\eta}\|A(x_0)^{-1}(F(x_n) + G(x_n))\| \leq 1$,
(d) $\|x_{n+1} - x_n\| \leq (t_{n+1} - t_n)\frac{\|x_n - x_{n-1}\|}{t_n - t_{n-1}}$,
(e) $\frac{\|A(x_0)^{-1}(F(x_n)+G(x_n))\|}{\|A(x_0)^{-1}(F(x_{n-1})+G(x_{n-1}))\|} \leq \frac{\varphi_\gamma(t_n)}{\varphi_\gamma(t_{n-1})}$ *for all $n \in \mathbb{N}_0$,*

where $\{t_n\}$ is a real sequence defined by (37).

Theorem 4.6. *Assume that all the assumptions of Lemma 4.5 hold and*

$$\beta \leq \min\left\{\frac{1}{\sqrt{\eta}}, \frac{b + 2c - 2\sqrt{c(c+b)}}{\gamma(1+\sqrt{\eta})}\right\}.$$

Then the sequence $\{x_n\}$ generated by Inexact Newton-like Algorithm 1.1 is well defined, remains in $B_{t^*}[x_0]$ and converges to the solution $x^* \in B_{t^*}[x_0]$ of (8). Moreover, the following error estimate hold:

$$\|x_{n+1} - x_n\| \leq \theta^{2^{n-1}}\|x_n - x_{n-1}\|,$$

$$\|x^* - x_n\| \leq \frac{\theta^{2^{n-1}}}{1 - \theta^{2^n}}\|x_1 - x_0\|,$$

$$\|A(x_0)^{-1}(F(x_n) + G(x_n))\| \leq \theta^{2^{n-1}}\|A(x_0)^{-1}(F(x_{n-1}) + G(x_{n-1}))\|, n \geq 1.$$

If we take $A(x) = F'_x$ for each $x \in D$ in Theorem 4.6, we have the following result:

Corollary 4.7. *Let X and Y be two Banach spaces, D a nonempty open convex subset of X and $x_0 \in D$. Let F: $D \to Y$ be a nonlinear Fréchet differentiable operator such that $F'^{-1}_{x_0}$ exists and G: $D \to Y$ a continuous operator. Assume that all the assumptions of Lemma 4.1 and (4.6) hold. We also assume that the γ-Lipschitz condition (39) and (10), (C1), (C3), (C5), (C6) hold on $B_{t^*}[x_0] \subseteq B_(r^*)(x_0)$ for $A(x) = F'_x$. Let $\lambda = (1 + \sqrt{\eta})\beta, b = 1 - (1 + \sqrt{\eta})K$ and $c = 1 + \sqrt{\eta} + \frac{\sqrt{2}\eta(1+\sqrt{\eta})}{\gamma(1-\sqrt{\eta})^2}$. Then the sequence $\{x_n\}$ generated by Inexact Newton-like Algorithm 1.1 is well defined, remains in $B_{t^*}[x_0]$ and converges to the solution $x^* \in B_{t^*}[x_0]$ of equation (8). Moreover, the following error estimate hold:*

$$\|x_{n+1} - x_n\| \leq \theta^{2^{n-1}}\|x_n - x_{n-1}\|, \|x^* - x_n\| \leq \frac{\theta^{2^{n-1}}}{1 - \theta^{2^n}}\|x_1 - x_0\| \text{ and}$$

$$\|F'^{-1}_{x_0}(F(x_n) + G(x_n))\| \leq \theta^{2^{n-1}}\|F'^{-1}_{x_0}(F(x_{n-1}) + G(x_{n-1}))\|, n \geq 1.$$

If we take $A(x) = F'_x$ and $G(x) = 0$ for each $x \in D$ in Theorem 4.6, we have the following result:

Corollary 4.8. *Let X and Y be two Banach spaces, D a nonempty open convex subset of X and $x_0 \in D$. Let F: $D \to Y$ be a nonlinear Fréchet differentiable operator such that $F'^{-1}_{x_0}$ exists. Assume that all the assumptions of Lemma 4.1 hold. We also assume that the γ-Lipschitz condition (39), and (10), (C1), (C5), (C6) hold on $B_{t^*}[x_0] \subseteq B_{r^*}(x_0)$ for $A(x) = F'_x$ and $G(x) = 0$. Let $\lambda = (1 + \sqrt{\eta})\beta, b = 1, c = 1 + \sqrt{\eta} + \frac{\sqrt{2}\eta(1+\sqrt{\eta})}{\gamma(1-\sqrt{\eta})^2}$ and $\beta \leq \min\{\frac{1}{\sqrt{\eta}}, \frac{1+2c-2\sqrt{c(c+1)}}{\gamma(1+\sqrt{\eta})}\}$. Then the sequence $\{x_n\}$ generated by Inexact Newton-like Algorithm 1.1 is well defined, remains in $B_{t^*}[x_0]$ and converges to the solution $x^* \in B_{t^*}[x_0]$ of $F(x) = 0$. Moreover, the following error estimate hold:*

$$\|x_{n+1} - x_n\| \leq \theta^{2^{n-1}}\|x_n - x_{n-1}\|, \|x^* - x_n\| \leq \frac{\theta^{2^{n-1}}}{1 - \theta^{2^n}}\|x_1 - x_0\| \text{ and}$$

$$\|F'^{-1}_{x_0}F(x_n)\| \leq \theta^{2^{n-1}}\|F'^{-1}_{x_0}F(x_{n-1})\|, n \geq 1.$$

Remark 4.9. For the choice of $G = 0$, Corollary 3.8 reduces to the well known Newton Kantorovich theorem which was already discussed by Wang [4], Tapia [35].

Remark 4.10. If $A(x) = F'_x, G(x) = 0$ for each $x \in D \subseteq X$ and $L_0(t) = L(t)$ for each $t \in [0, r]$, then Theorem 3.4 reduces to [29, Theorem 3.1].

Remark 4.11. If $A(x) = F'_x, G(x) = 0$ for each $x \in D \subseteq X$ and $L_0(t) \leq L(t)$ for each $t \in [0, r]$, then Theorem 3.4 reduces to [33, Theorem 2.2].

Remark 4.12. If $A(x) = F'_x$ and $G(x) = 0$ for each $x \in D \subseteq X$ and $\gamma_0 = \gamma$, then Corollary 4.8 reduces to [28, Theorem 3.1].

Remark 4.13. For an analytic operator F, we can define γ by

$$\gamma = \sup_{n>1} \left\{ \frac{\|F_{x_0}'^{-1} F_{x_0}^{(n)}\|}{n!} \right\}^{\frac{1}{n-1}}.$$

This definition is due to Smale [3], (see also [8,28,32,36]).

In the following example, we assume that $A(x) = F'_x$ for each $x \in D$ and $\lambda = \lambda_0$.

Example 4.14. Let $D = X = Y = C([0, 1])$ be the space of all continuous real-valued functions defined on [0,1] with norm $\|x\| = \sup\{|x(t)| : t \in [0, 1]\}$. Consider the following Fredholm integral equation:

$$x(s) = -\frac{|x(s)|}{12\pi} - \frac{\sin(\pi s)}{4} + \frac{\cos(\pi s)}{12} \int_0^1 \sin(\pi t) x^2(t) dt.$$

Define F, G: $D \to Y$ by

$$F(x) = x(s) + \frac{\sin(\pi s)}{4} - \frac{\cos(\pi s)}{12} \int_0^1 \sin(\pi t) x^2(t) dt,$$

$$G(x) = \frac{|x(s)|}{12\pi} - \frac{\sin(\pi s)}{2}$$

and

$$F(x) + G(x) = x(s) + \frac{|x(s)|}{12\pi} - \frac{\sin(\pi s)}{4} - \frac{\cos(\pi s)}{12} \int_0^1 \sin(\pi t) x^2(t) dt. \tag{40}$$

The Fréchet derivative of F at point $x \in C([0, 1])$ is

$$F'_x(h(s)) = h(s) - \frac{\cos(\pi s)}{6} \int_0^1 \sin(\pi t) x(t) h(t) dt \text{ for each } h \in C([0, 1]).$$

For $x_0(s) = 0$, we have the following

$$F_{x_0}'^{-1} h(s) = h(s) \text{ for each } h \in C([0, 1]), \|F_{x_0}'^{-1}(F(x_0) + G(x_0))\| \leq \frac{1}{4}$$

and

$$\|F_{x_0}'^{-1}(G(x) - G(y))\| \leq \frac{1}{6\pi} \|x - y\| \text{ for each } x, y \in C([0, 1]).$$

Thus $K = \frac{1}{12\pi}$, $\gamma = \frac{1}{6\pi}$, $\beta = \frac{1}{4}$, $\delta = 0$ and for the choice of $\eta = 0.09$, we have $\lambda = 0.325$, $\kappa = \frac{1}{12\pi}$, $b = 0.96551642899676$, $c = 7.66510984201464$, $\lambda\gamma = 0.01724178550162$, $b + 2c - 2\sqrt{c(c + b)} = 0.02862839558843$ clearly $\lambda\gamma \leq b + 2c - 2\sqrt{c(c + b)}$. Thus function $\varphi'_\gamma(t)$ has zero $r^* = 1.08562159203139$ and function $\varphi_\gamma(t)$ has zero $t^* = 0.31798701471680$ and $\theta = 0.21805507285827$. Now

$$\int_0^{\|x - x_0\|} L(u) du \leq \int_0^{r^*} L(u) du \leq \frac{1}{(1 - \gamma r^*)^2} - 1 = 0.12596250398194 < 1$$

and $\sqrt{\eta}\|F_{x_0}'^{-1}(F(x_0) + G(x_0))\| \leq 0.075 < 1$. Since all the assumption of Corollary 4.7 hold. Therefore, the sequence $\{x_n\}$ generated by Algorithm 1.1 is well defined, remains in $B_{t^*}[x_0] \subseteq B_{r^*}(x_0)$ and converges to the solution $x^* \in B_{t^*}[x_0]$ of equation $F(x) + G(x) = 0$, where $F(x) + G(x)$ is defined in (40).

Remark 4.15. In Example 4.14, we cannot apply Theorem [32, Theorem 2.9] and [28, Theorem 3.1] because $F(x) + G(x)$ is not Fréchet differentiable on $C([0, 1])$. However all the conditions of Corollary 4.7 are satisfied.

5. Conclusions

We extend and improve the inexact Newton method [28,29,32,33] in the context of differentiability of involved operator and introduced inexact Newton-like algorithm. We have discussed the semilocal convergence analysis of our algorithm under the average Lipschitz condition as well as γ-Lipschitz condition. We have also provided an example which shows the effectiveness of our result.

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