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RESEARCH ARTICLE

An inertial iterative method for common solution of a pair of variational inequality problems

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ABSTRACT
In this paper, we have presented a novel inertial-viscosity approximation technique for finding the common element (CE) of solution set of two variational inequality problems in the real Hilbert spaces. First, we investigate the CE of the set of fixed points (FPs) of two Nonexpansive (NE) mappings and then apply our result to find the CE of the solution set of two variational inequality problems. We also discussed on the strong convergence of the proposed algorithm under some moderate circumstances. Our result improves, extends and generalizes some published results in this direction. Furthermore, our results are supported by some numerical examples.

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1. Introduction

This article assumes that $\mathcal{H}$ is a real Hilbert space with norm $\|.\|$ and inner product $\langle .,. \rangle$. The symbol $\mathbb{N}$ stands for the set of all natural numbers. Let E be a closed convex subset (CCS) of $\mathcal{H}$ that is not empty. The variational inequality problem (VIP) for mapping $A : E \to \mathcal{H}$ is formulated as

$$\text{to find} \quad \mu^* \in E \quad \text{such that} \quad \langle A\mu^*, \mu - \mu^* \rangle \geq 0 \quad \text{for each} \quad \mu \in E. \tag{1}$$

The VIP solution set is represented by Ω_A, i.e.

$$\Omega_A = \{\mu^* \in E : \langle A\mu^*, \mu - \mu^* \rangle \geq 0 \text{ for each } \mu \in E\}.$$

Assume a mapping $R : E \to E$. The notation $Fix(R)$ represents the set of FPs of R, i.e.

$$Fix(R) = \{\mu \in E : \mu = R(\mu)\}.$$

One of the core issues in optimization theory is figuring out how to solve VIP. In recent years, academics have typically employed two methods: one is the projection method and another is the regularized method to solve the monotone variational inequality (MVI) problem under suitable circumstances. Applications for these MVI are numerous and include resource allocation, network equilibrium flow, picture restoration, optimal control, signal processing, engineering design optimization difficulties and more [24,38,39,41]. Numerous approaches to solve the VIP have been researched recently.

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In 2000, Maudafi [25] introduced a viscosity approximation method (VAM) for solving the VIP in $\mathcal{H}$. Let us consider the following VIP:

$$\text{to find } \mu^* \in Fix(R) \quad \text{such that} \quad \langle (I-\Theta)(\mu^*), \mu - \mu^* \rangle \geq 0 \quad \text{for all } \mu \in Fix(R), \tag{2}$$

where R is an NE mapping and Θ is a contraction mapping. To solve the VIP (2), the VAM was given as follows:

$$\begin{cases} \mu_1 \in E, \\ \mu_{j+1} = \dfrac{\varsigma_j}{1+\varsigma_j}\Theta(\mu_j) + \dfrac{1}{1+\varsigma_j}R(\mu_j) \quad \text{for all } j \in \mathbb{N}, \end{cases} \tag{3}$$

where the sequence $\{\varsigma_j\}$ in $(0,1)$ satisfies the criteria mentioned below:

(i) $\lim_{j\to\infty} \varsigma_j = 0$ and $\sum_{j=1}^{\infty} \varsigma_j = \infty$;
(ii) $\lim_{j\to\infty}(\frac{1}{\varsigma_j} - \frac{1}{\varsigma_{j+1}}) = 0$.

Moudafi [25] proved the sequence $\{\mu_j\}$ created by (3) converges to the solution $\mu^* \in Fix(R)$ strongly of the VIP (2). In 2004, Xu [43] introduced an iteration method to obtain the solution of VIP and FPs of an NE function defined as follows:

$$\begin{cases} \mu_1 \in E, \\ \mu_{j+1} = \varsigma_j \Theta(\mu_j) + (1-\varsigma_j)R(\mu_j), \quad j \in \mathbb{N}, \end{cases} \tag{4}$$

where the sequence $\{\varsigma_j\}$ in $(0,1)$ satisfies the criteria mentioned below:

(i) $\varsigma_j \to 0$, $\sum_{j=1}^{\infty} \varsigma_j = \infty$;
(ii) $\sum_{j=1}^{\infty} |\varsigma_{j+1} - \varsigma_j| < \infty$ or $\lim_{j\to\infty} \frac{\varsigma_{j+1}}{\varsigma_j} = 1$.

Furthermore, it was demonstrated that $\{\mu_j\}$ produced by (4) converges to the one and only solution $\mu^* \in Fix(R)$ strongly of the VIP (2). Several researchers presented many iterative techniques for identifying the CE of the set of operator's FPs and the solutions set of VIP and discussed their convergence analysis. In this sequel, Takahashi et al. [37] presented an iteration procedure as follows:

$$\begin{cases} \mu_1 \in E, \\ \mu_{j+1} = \varsigma_j \mu_j + (1-\varsigma_j)RP_E(\mu_j - \lambda_j A\mu_j) \quad \text{for all } j \in \mathbb{N}, \end{cases} \tag{5}$$

where $R: E \to E$ is the NE operator, P_E is the metric projection mapping and A is the inverse strongly-monotone (ISM) operator and weak convergence of (5) was established under some modest conditions. Inspired by Takahashi et al. study [37], Iiduka and Takahashi [16] introduced a new iterative method:

$$\begin{cases} \mu_1 \in E, \\ \mu_{j+1} = \varsigma_j \mu_1 + (1-\varsigma_j)RP_E(\mu_j - \lambda_j A\mu_j) \quad \text{for all } j \in \mathbb{N}, \end{cases} \tag{6}$$

where $A: E \to \mathcal{H}$ is the α-ISM mapping, $R: E \to E$ is the NE operator, the sequence $\{\varsigma_j\}$ in $[0,1)$, and the sequence $\{\lambda_j\}$ in $[a,b]$ for some a, b with $0 < a < b < 2\alpha$ and demonstrated strong convergence of (6). In [9], Chen et al. introduced a VAM:

$$\begin{cases} \mu_1 \in E, \\ \mu_{j+1} = \varsigma_j \Theta(\mu_j) + (1-\varsigma_j)RP_E(\mu_j - \lambda_j A\mu_j) \quad \text{for all } j \in \mathbb{N}, \end{cases} \tag{7}$$

where $R: E \to E$ is an NE mapping and $\Theta: E \to E$ is a contraction mapping. They also discussed its strong convergence analysis under suitable conditions.

Sahu et al. presented an iterative method in [32] for identifying the CE of the set of FPs of an operator and the set of solutions to the VIPs given as follows. For any $\mu_1 \in E$,

$$\begin{cases} \mu_{j+1} = \varrho_j SP_E(\mu_j - s_j A\mu_j) + (1-\varrho_j)SP_E(\nu_j - t_j B\nu_j); \\ \nu_j = \varsigma_j \Theta(\mu_j) + (1-\varsigma_j)\mu_j \quad \text{for all } j \in \mathbb{N}, \end{cases} \tag{8}$$

where $S : E \to E$ is the NE, $\Theta : E \to E$ is a contraction mapping and $A, B : E \to \mathcal{H}$ are two ISM mappings. Under suitable conditions of iteration parameters $\{\varsigma_j\}$, $\{\varrho_j\}$, $\{s_j\}$ and $\{t_j\}$, they established the strong convergence of algorithm (8).

On the other hand, the S-iteration technique, which is an FP iterative technique was presented by Agarwal et al. [1] in 2007 for computing the FPs of nonlinear operators. The process is defined as follows: for any $\mu_1 \in E$,

$$\begin{cases} \mu_{j+1} = (1-\varsigma_j)R(\mu_j) + \varsigma_j R(\nu_j); \\ \nu_j = (1-\varrho_j)\mu_j + \varrho_j R(\mu_j), \quad j \in \mathbb{N}, \end{cases} \tag{9}$$

where the sequences $\{\varsigma_j\}$ and $\{\varrho_j\}$ are in $(0, 1)$, $R : E \to E$ is an operator and E is a CCS of a Banach space. Further, many authors extended several varieties of S-iteration methods in different spaces [2,14,15,17–21,30,31,36,44].

In 2010, Sahu [31] proposed a new technique which is called Halpern S-iteration method (HSIM) given as follows. For any $\mu_1 \in E$,

$$\begin{cases} \nu_j = (1-\varsigma_j)\mu_j + \varsigma_j u; \\ \mu_{j+1} = (1-\varrho_j)R(\mu_j) + \varrho_j R(\nu_j) \quad \text{for all } j \in \mathbb{N}, \end{cases} \tag{10}$$

where $u \in E$ and the sequences $\{\varsigma_j\}$, $\{\varrho_j\}$ are in $(0, 1)$ satisfying the criteria mentioned below:

(i) $\varsigma_j \to 0, \sum_{j=1}^{\infty} \varsigma_j \varrho_j = \infty, \frac{\varsigma_j}{\varsigma_{j+1}} \to 1, \frac{\varrho_j}{\varrho_{j+1}} \to 1.$

He demonstrated the proposed algorithm's strong convergence (10).

In [29], Qin et al. inspired by work of Kim et al. [22] and Qin et al. [28] proposed the new iteration process and under some moderate circumstances they proved that strong convergence of proposed algorithm, given as follows. For any $\mu_1 \in E$,

$$\begin{cases} \nu_j = \varrho_j S(\mu_j) + (1-\varrho_j)R(\mu_j); \\ \mu_{j+1} = \varsigma_j u + (1-\varsigma_j)\nu_j \quad j \in \mathbb{N}, \end{cases} \tag{11}$$

where the sequences $\{\varsigma_j\}$ and $\{\varrho_j\}$ are in $(0, 1)$ such that

(i) $\varsigma_j \to 0, \varrho_j \to \varrho \in (0, 1)$;
(ii) $\sum_{j=1}^{\infty} \varsigma_j = \infty$;
(iii) $\sum_{j=1}^{\infty} |\varsigma_{j+1} - \varsigma_j| < \infty$;
(iv) $\sum_{j=1}^{\infty} |\varrho_{j+1} - \varrho_j| < \infty$.

Common FPs of two operators can be computed thanks to an iterative technique called S-iteration of rank 3, which was recently introduced by Sahu et al. [33]. Under moderate iteration parameter conditions, they demonstrated the suggested algorithm's weak convergence. They offered a few innovative ways to locate the zero of accretive operators, convex programming problems and split feasibility problems.

However, Polyak [27] initially suggested the heavy ball approach of the two-order temporal dynamical system to speed up the convergence rate of algorithms for minimizing a smooth convex function. In further efforts to enhance the convergence rate of heavy ball method, Nesterov [26] presented a modified heavy ball method. For any $\mu_0, \mu_1 \in \mathcal{H}$,

$$\begin{cases} \nu_j = \mu_j + \theta_j(\mu_j - \mu_{j-1}); \\ \mu_{j+1} = \nu_j - \lambda_j \nabla f(\nu_j), \quad j \geq 1, \end{cases}$$

where $\{\lambda_j\}$ is a positive sequence of real numbers and $\theta_j \in [0, 1)$ is an extrapolation factor. Alvarez and Attouch [4] presented the inertial proximal point algorithm in 2001 as a method for utilizing the proximal point algorithm to get the zero point of a maximum monotone operator.

In [23], Maingé applied the inertial technique to the Mann iteration process and presented an inertial Mann iteration (IMI) $\{\mu_j\}$ for evaluating the FPP of NE mappings in $\mathcal{H}$, which is defined as follows. For any $\mu_0, \mu_1 \in \mathcal{H}$,

$$\begin{cases} \nu_j = \mu_j + \theta_j(\mu_j - \mu_{j-1}); \\ \mu_{j+1} = \nu_j + \varsigma_j(R(\nu_j) - \nu_j), \quad j \geq 1, \end{cases} \tag{12}$$

where R is the NE mapping on $\mathcal{H}$, $\theta \in [0, 1)$, $\theta_j \in [0, \theta)$ and $\varsigma_j \in (0, 1)$. In the following situations, the author demonstrated that the sequence $\{\mu_j\}$ converges weakly to an FP of R::

(i) $\sum_{j=1}^{\infty} \theta_j \|\mu_j - \mu_{j-1}\|^2 < \infty$;
(ii) $0 < \inf \varsigma_j \leq \sup \varsigma_j < 1$ for all $j \geq 1$.

The classical IMI process was presented by Bot et al. in [7]. They replaced the Maingé [23] criteria with alternative ones and demonstrated the weak convergence of the classical IMI. Dong et al. [13] developed the general IMI with additional criteria and demonstrated the weak convergence of the suggested algorithm, further relaxing the conditions of Maingé [23] and Bot et al. [7]. Additionally, the IMI approach (12) for computing the FPs of R with affine domain in $\mathcal{H}$ was examined for weak convergence by Sahu [34]. The recent results on inertial techniques have been studied by several researchers [11–13,35,40].

Many authors introduced different methods to find the common solution of FPP and VIP [16,32,37,38]. Inspired by these works we introduce a novel method to find the common solution of a pair of VIPs. For finding the common solution (CS) of a pair of VIP is formulated as

$$\text{to find} \quad \mu^* \in E \quad \text{such that} \quad \mu^* \in \Omega_A \cap \Omega_B = \Omega,$$

where $A, B : E \to \mathcal{H}$ are two mappings.

Raising the following question is ordinary.

Question 1.1: Is it possible to develop a novel iteration process for finding the CS of a pair of VIP?

Inspired and based on works of Sahu et al. [31–34], Qin et al. [29], Chen et al. [9] and Bot et al. [7], we propose a novel inertial-viscosity approximation technique for finding the CE of solution sets of two variational inequality problems. The aim of this paper is to investigate the common FP of two NE mappings. Then we apply our result for finding the CS of a pair of VIP. Our result is improvement over the well-established results of Qin et al. [29, Theorem 2.1] and Sahu [31, Theorem 5.4]. Under certain mild conditions, we prove that the sequence $\{\mu_j\}$ produced by the proposed algorithm converge strongly to a CE of FP sets of operators S and R, which is unique solution of VIP (2). Our result provides affirmative answer of Question 1.1.

The structure of this paper is as follows: we included certain fundamental concepts and lemmas in Section 2 that are used to analyse our findings. We describe an approach for approximating inertial viscosity and its convergence theory in Section 3. With the use of a table and figures, we compare our findings and provide some numerical examples in Section 4. In Section 5, we finally concluded up our results.

2. Preliminaries

The strong (or weak) convergence of a sequence $\{\mu_j\}$ to μ will be denoted by $\mu_j \to \mu$ (or $mu_j \rightharpoonup \mu$). Let E be a nonempty subset of $\mathcal{H}$, $R, S : E \to E$ are two mappings and $\Gamma = Fix(S) \cap Fix(R)$ denote the set of common FPs of mappings R and S. A mapping $R : E \to E$ is called:

(i) nonexpansive if

$$\|R(\mu) - R(\nu)\| \leq \|\mu - \nu\| \quad \forall\, \mu, \nu \in E;$$

(ii) monotone if

$$\langle R\mu - T\nu, \mu - \nu\rangle \geq 0 \quad \forall\, \mu, \nu \in E;$$

(iii) η-strongly monotone if

$$\langle R\mu - R\nu, \mu - \nu\rangle \geq \eta\|\mu - \nu\|^2 \;\; \forall\, \mu, \nu \in E;$$

for some $\eta > 0$

(iv) α-ISM

$$\langle R\mu - R\nu, \mu - \nu\rangle \geq \alpha\|R\mu - R\nu\|^2 \quad \text{for all } \mu, \nu \in E$$

for some $\alpha > 0$.

The mapping $\Theta : E \to E$ is called κ-contraction if

$$\|\Theta(\mu) - \Theta(\nu)\| \leq \kappa\|\mu - \nu\| \quad \text{for each } \mu, \nu \in E,$$

where $\kappa \in [0, 1)$. Let E be a nonempty CCS of $\mathcal{H}$. Then, for any $\mu \in \mathcal{H}$, there exists a unique nearest point $P_E(\mu)$ of E such that

$$\|\mu - P_E(\mu)\| \leq \|\mu - \nu\| \quad \text{for all } \nu \in E.$$

The metric projection from $\mathcal{H}$ onto E is the mapping P_E. Keep in mind that, according to [31], the metric projection mapping P_E is NE from $\mathcal{H}$ onto E.

Lemma 2.1 ([31]): *Let E be a nonempty CCS of $\mathcal{H}$. Given $\mu \in \mathcal{H}$ and $\omega \in E$. Then $\omega = P_E(\mu) \Longleftrightarrow \langle \mu - \omega, \omega - \nu\rangle \geq 0$ for each $\nu \in E$.*

Lemma 2.2 ([42, Lemma 2.1]): *Let $\{\varsigma_j\}$ and $\{\varrho_j\}$ be two sequences of real numbers such that:*

(i) $\{\varsigma_j\} \subset [0, 1]$ *and* $\sum_{j=0}^{\infty} \varsigma_j = \infty$ *or equivalently* $\prod_{j=0}^{\infty}(1 - \varsigma_j) = 0$,
(ii) $\limsup_{j\to\infty} \varrho_j \leq 0$ *or*
(iii) $\sum_{j=0}^{\infty} \varsigma_j\varrho_j$ *is convergent.*

If $\{s_j\}$ is a sequence of real numbers that are not negative such that

$$s_{j+1} \leq (1 - \varsigma_j)s_j + \varsigma_j\varrho_j \quad \forall\, j \geq 0.$$

Then $\lim_{j\to\infty} s_j = 0$.

Lemma 2.3 ([8]): *Let $\mathcal{H}$ be an inner product space, then*

$$\|\mu+\nu\|^2 \leq \|\mu\|^2 + 2\langle \nu, \mu+\nu\rangle \quad \forall\, \mu,\nu \in \mathcal{H}.$$

Lemma 2.4 ([6]): *Let E be a nonempty CCS of $\mathcal{H}$ and $R : E \to \mathcal{H}$ be an NE mapping. Let $\{\mu_j\}$ be a sequence in E and $\mu \in \mathcal{H}$ such that $\mu_j \rightharpoonup \mu$ and $\mu_j - R(\mu_j) \to 0$ as $j \to \infty$. Then $\mu \in Fix(R) = \{\omega \in E : R(\omega) = \omega\}$.*

Lemma 2.5 ([6]): *Let $\mathcal{H}$ be a real Hilbert space. Let $\mu,\nu \in \mathcal{H}$ and $t \in \mathbb{R}$. Then*

$$\|t\mu + (1-t)\nu\|^2 = t\|\mu\|^2 + (1-t)\|\nu\|^2 - t(1-t)\|\mu-\nu\|^2.$$

Lemma 2.6 ([10]): *Let A be a bounded linear operator from $\mathcal{H}$ into itself with the adjoint A^* and let $b \in \mathcal{H}$. Let E be a nonempty CCS of $\mathcal{H}$. Let $\omega \in \mathcal{H}$ and $\gamma \in (0,\infty)$. Then the following claims are interchangeable:*

(i) *The following problem is resolved by ω:*

$$\min_{\mu\in E} \frac{1}{2}\|A\mu - b\|^2.$$

(ii) $\omega = P_E(\omega - \gamma A^*(A\omega - b))$.
(iii) $\langle A\nu - A\omega, b - A\omega\rangle \leq 0$ *for all* $\nu \in E$.

Lemma 2.7 ([3, Theorem 5.3.1]): *Let E be a nonempty CCS of $\mathcal{H}$ and $A : \mathcal{H} \to \mathcal{H}$ be a given operator. Then the following claims are interchangeable:*

(a) $\mu^* \in \Omega_A$,
(b) $\mu^* \in Fix(P_E(I - \lambda A))$ *for any* $\lambda > 0$.

3. Main results

Algorithm 1

Let E be a nonempty closed affine subset of $\mathcal{H}$ and let $R, S : E \to E$ be two NE mappings such that $\Gamma := Fix(S) \cap Fix(R) \neq \emptyset$. Assume that $\Theta : E \to E$ is a κ-contraction mapping.
Initialization: Select arbitrary starting points $\mu_0, \mu_1 \in E$ and set $j = 1$.
Iterative step: Given the current iterates μ_{j-1} and μ_j, for $j \geq 1$ calculate the next iteration as follows:

$$\begin{cases} \nu_j = \mu_j + \theta_j(\mu_j - \mu_{j-1}), \\ \omega_j = \varsigma_j\Theta(\nu_j) + (1-\varsigma_j)\nu_j, \\ \mu_{j+1} = \varrho_j S(\nu_j) + (1-\varrho_j)R(\omega_j), \quad j \in \mathbb{N}, \end{cases} \tag{13}$$

where $\{\varsigma_j\}$, $\{\varrho_j\}$ and $\{\theta_j\}$ are real sequences which fulfils the requirements:

($\curlywedge_1$) $\{\varsigma_j\} \subset (0,1)$ for all $j \in \mathbb{N}$, $\lim_{j\to\infty} \varsigma_j = 0$, $\sum_{j=1}^{\infty} \varsigma_j = \infty$ and $\frac{\varsigma_j}{\varsigma_{j+1}} \to 1$,
($\curlywedge_2$) $0 < a \leq \varrho_j \leq b < 1, j \in \mathbb{N}$ and $\frac{|\varrho_{j+1}-\varrho_j|}{\varsigma_j} \to 0$,
($\curlywedge_3$) $\frac{\theta_j}{\varsigma_j}\|\mu_j - \mu_{j-1}\| \to 0$ as $n \to \infty$, $\theta_j \in [0,\theta], 0 \leq \theta < 1$.

Theorem 3.1: *Let E be a nonempty closed affine subset of $\mathcal{H}$ and let $R, S : E \to \mathcal{H}$ be two NE mappings such that $\Gamma := Fix(S) \cap Fix(R) \neq \emptyset$. Let $\Theta : E \to E$ be a κ-contraction and $\{\varsigma_j\}$, $\{\varrho_j\}$, $\{\theta_j\}$ are real sequences which fulfils the requirements $(\curlywedge_1)$, $(\curlywedge_2)$ and $(\curlywedge_3)$. Then the sequence $\{\mu_j\}$ produced by Algorithm 3.1 converges to the unique solution $\Upsilon \in \Gamma$ strongly of the VIP:*

$$\langle (I - \Theta)\Upsilon, \mu - \Upsilon \rangle \geq 0 \quad \text{for all } \mu \in \Gamma.$$

Proof: Let $\Upsilon \in \Gamma$. From (13), we have

$$\begin{aligned} \|\nu_j - \Upsilon\| &= \|\mu_j + \theta_j(\mu_j - \mu_{j-1}) - \Upsilon\| \\ &\leq \|\mu_j - \Upsilon\| + \theta_j\|\mu_j - \mu_{j-1}\|. \end{aligned} \tag{14}$$

From (13) and (17), we have

$$\begin{aligned} \|\omega_j - \Upsilon\| &= \|(1 - \varsigma_j)\nu_j + \varsigma_j\Theta(\nu_j) - \Upsilon\| \\ &\leq (1 - \varsigma_j)\|\nu_j - \Upsilon\| + \varsigma_j\|\Theta(\nu_j) - \Upsilon\| \\ &\leq (1 - \varsigma_j)\|\nu_j - \Upsilon\| + \varsigma_j(\|\Theta(\nu_j) - \Theta(\Upsilon)\| + \|\Theta(\Upsilon) - \Upsilon\|) \\ &\leq (1 - \varsigma_j)\|\nu_j - \Upsilon\| + \varsigma_j(\kappa\|\nu_j - \Upsilon\| + \|\Theta(\Upsilon) - \Upsilon\|) \\ &= (1 - \varsigma_j(1 - \kappa))\|\nu_j - \Upsilon\| + \varsigma_j\|\Theta(\Upsilon) - \Upsilon\| \\ &\leq (1 - \varsigma_j(1 - \kappa))\|\mu_j - \Upsilon\| + (1 - \varsigma_j(1 - \kappa))\theta_j\|\mu_j - \mu_{j-1}\| + \varsigma_j\|\Theta(\Upsilon) - \Upsilon\|. \end{aligned} \tag{15}$$

From (13) and (15), we have

$$\begin{aligned} \|\mu_{j+1} - \Upsilon\| &\leq \varrho_j\|S(\nu_j) - \Upsilon\| + (1 - \varrho_j)\|R(\omega_j) - \Upsilon\| \\ &\leq \varrho_j\|\nu_j - \Upsilon\| + (1 - \varrho_j)\|\omega_j - \Upsilon\| \\ &\leq \varrho_j(\|\mu_j - \Upsilon\| + \theta_j\|\mu_j - \mu_{j-1}\|) + (1 - \varrho_j)(1 - \varsigma_j(1 - \kappa))\|\mu_j - \Upsilon\| \\ &\quad + (1 - \varrho_j)(1 - \varsigma_j(1 - \kappa))\theta_j\|\mu_j - \mu_{j-1}\| + \varsigma_j(1 - \varrho_j)\|\Theta(\Upsilon) - \Upsilon\| \\ &= (1 - \varsigma_j(1 - \varrho_j)(1 - \kappa))\|\mu_j - \Upsilon\| + \varsigma_j(1 - \varrho_j)\|\Theta(\Upsilon) - \Upsilon\| \\ &\quad + (1 - \varsigma_j(1 - \varrho_j)(1 - \kappa))\theta_j\|\mu_j - \mu_{j-1}\| \\ &= (1 - \varsigma_j(1 - \varrho_j)(1 - \kappa))\|\mu_j - \Upsilon\| + \varsigma_j(1 - \varrho_j)(1 - \kappa)\left(\frac{\|\Theta(\Upsilon) - \Upsilon\|}{1 - \kappa}\right. \\ &\quad \left. + \frac{(1 - \varsigma_j(1 - \varrho_j)(1 - \kappa))}{(1 - \varrho_j)(1 - \kappa)}\frac{\theta_j\|\mu_j - \mu_{j-1}\|}{\varsigma_j}\right). \end{aligned} \tag{16}$$

By the condition $\frac{\theta_j}{\varsigma_j}\|\mu_j - \mu_{j-1}\| \to 0$ as $j \to \infty$, there exists a constant $M \geq 0$ such that $(1 - \varsigma_j(1 - \varrho_j)(1 - \kappa))\frac{\theta_j\|\mu_j - \mu_{j-1}\|}{\varsigma_j(1-\varrho_j)} \leq M$ for all $j \in \mathbb{N}$. Hence

$$\begin{aligned} \|\mu_{j+1} - \Upsilon\| &\leq (1 - \varsigma_j(1 - \varrho_j)(1 - \kappa))\|\mu_j - \Upsilon\| + \varsigma_j(1 - \varrho_j)(1 - \kappa)\left(\frac{M + \|\Theta(\Upsilon) - \Upsilon\|}{1 - \kappa}\right) \\ &\leq \max\left\{\|\mu_j - \Upsilon\|, \frac{M + \|\Theta(\Upsilon) - \Upsilon\|}{1 - \kappa}\right\} \\ &\;\;\vdots \\ &\leq \max\left\{\|\mu_1 - \Upsilon\|, \frac{M + \|\Theta(\Upsilon) - \Upsilon\|}{1 - \kappa}\right\}. \end{aligned} \tag{17}$$

This demonstrates that the sequence $\{\mu_j\}$ is bounded, and it is simple to demonstrate that the sequences $\{\nu_j\}$ and $\{\omega_j\}$ are bounded as well using (13).

From (13), $(\curlywedge_1)$ and $(\curlywedge_3)$, we have

$$\|\nu_j - \mu_j\| = \theta_j\|\mu_j - \mu_{j-1}\| \leq \frac{\theta_j}{\varsigma_j}\|\mu_j - \mu_{j-1}\|\varsigma_j \to 0 \quad \text{as } j \to \infty \tag{18}$$

and

$$\|\omega_j - \nu_j\| = \varsigma_j\|\Theta(\nu_j) - \nu_j\| \to 0 \quad \text{as } j \to \infty. \tag{19}$$

Using (13), we have

$$\begin{aligned}\|\nu_j - \nu_{j-1}\| &= \|\mu_j + \theta_j(\mu_j - \mu_{j-1}) - \mu_{j-1} - \theta_{j-1}(\mu_{j-1} - \mu_{j-2})\| \\ &\leq \|\mu_j - \mu_{j-1}\| + \theta_j\|\mu_j - \mu_{j-1}\| + \theta_{j-1}\|\mu_{j-1} - \mu_{j-2}\|\end{aligned} \tag{20}$$

and

$$\begin{aligned}\|\omega_j - \omega_{j-1}\| &= \|\varsigma_j\Theta(\nu_j) + (1-\varsigma_j)\nu_j - \varsigma_{j-1}\Theta(\nu_{j-1}) - (1-\varsigma_{j-1})\nu_{j-1}\| \\ &= \|\varsigma_j\Theta(\nu_j) - \varsigma_j\Theta(\nu_{j-1}) + \varsigma_j\Theta(\nu_{j-1}) + (1-\varsigma_j)\nu_j - (1-\varsigma_j)\nu_{j-1} \\ &\quad + (1-\varsigma_j)\nu_{j-1} - \varsigma_{j-1}\Theta(\nu_{j-1}) - (1-\varsigma_{j-1})\nu_{j-1}\| \\ &\leq \varsigma_j\|\Theta(\nu_j) - \Theta(\nu_{j-1})\| + (1-\varsigma_j)\|\nu_j - \nu_{j-1}\| \\ &\quad + |\varsigma_j - \varsigma_{j-1}|\|\Theta(\nu_{j-1}) - \nu_{j-1}\| \\ &\leq \varsigma_j\kappa\|\nu_j - \nu_{j-1}\| + (1-\varsigma_j)\|\nu_j - \nu_{j-1}\| \\ &\quad + |\varsigma_j - \varsigma_{j-1}|\|\Theta(\nu_{j-1}) - \nu_{j-1}\| \\ &= (1 - \varsigma_j(1-\kappa))\|\nu_j - \nu_{j-1}\| + |\varsigma_j - \varsigma_{j-1}|\|\Theta(\nu_{j-1}) - \nu_{j-1}\|.\end{aligned} \tag{21}$$

From (13), (20) and (21), we have

$$\begin{aligned}\|\mu_{j+1} - \mu_j\| &= \|\varrho_j S(\nu_j) + (1-\varrho_j)R(\omega_j) - \varrho_{j-1}S(\nu_{j-1}) - (1-\varrho_{j-1})R(\omega_{j-1})\| \\ &= \|\varrho_j(S(\nu_j) - S(\nu_{j-1})) + (1-\varrho_j)(R(\omega_j) - R(\omega_{j-1})) \\ &\quad + (\varrho_j - \varrho_{j-1})S(\nu_{j-1}) - (\varrho_j - \varrho_{j-1})R(\omega_{j-1})\| \\ &\leq \varrho_j\|S(\nu_j) - S(\nu_{j-1})\| + (1-\varrho_j)\|R(\omega_j) - R(\omega_{j-1})\| \\ &\quad + |\varrho_j - \varrho_{j-1}|\|S(\nu_{j-1}) - R(\omega_{j-1})\| \\ &\leq \varrho_j\|\nu_j - \nu_{j-1}\| + (1-\varrho_j)\|\omega_j - \omega_{j-1}\| \\ &\quad + |\varrho_j - \varrho_{j-1}|\|S(\nu_{j-1}) - \Upsilon + \Upsilon - R(\omega_{j-1})\| \\ &\leq \varrho_j\|\nu_j - \nu_{j-1}\| + (1-\varrho_j)\|\omega_j - \omega_{j-1}\| + |\varrho_j - \varrho_{j-1}|M_1 \\ &\leq \varrho_j\|\nu_j - \nu_{j-1}\| + (1-\varrho_j)((1-\varsigma_j(1-\kappa))\|\nu_j - \nu_{j-1}\| \\ &\quad + |\varsigma_j - \varsigma_{j-1}|\|\Theta(\nu_{j-1}) - \nu_{j-1}\|) + |\varrho_j - \varrho_{j-1}|M_1 \\ &= (1 - \varsigma_j(1-\varrho_j)(1-\kappa))\|\nu_j - \nu_{j-1}\| + |\varrho_j - \varrho_{j-1}|M_1 \\ &\quad + (1-\varrho_j)|\varsigma_j - \varsigma_{j-1}|\|\Theta(\nu_{j-1}) - \nu_{j-1}\| \\ &\leq (1 - \varsigma_j(1-\varrho_j)(1-\kappa))(\|\mu_j - \mu_{j-1}\| + \theta_j\|\mu_j - \mu_{j-1}\| + \theta_{j-1}\|\mu_{j-1} - \mu_{j-2}\|) \\ &\quad + (1-\varrho_j)|\varsigma_j - a_{j-1}|\|\Theta(\nu_{j-1}) - \nu_{j-1}\| + |\varrho_j - \varrho_{j-1}|M_1 \\ &\leq (1 - \varsigma_j(1-\varrho_j)(1-\kappa))\|\mu_j - \mu_{j-1}\| + \theta_j\|\mu_j - \mu_{j-1}\| + \theta_{j-1}\|\mu_{j-1} - \mu_{j-2}\|\end{aligned}$$

$$
\begin{aligned}
&+ (1-\varrho_j)|\varsigma_j - a_{j-1}|L + |\varrho_j - \varrho_{j-1}|M_1 \\
&\leq (1-\delta_j)\|\mu_j - \mu_{j-1}\| + \delta_j \left(\frac{\theta_j \|\mu_j - \mu_{j-1}\|}{\varsigma_j(1-\varrho_j)(1-\kappa)} + \frac{\theta_{j-1}\|\mu_{j-1} - \mu_{j-2}\|}{\varsigma_j(1-\varrho_j)(1-\kappa)} \right. \\
&\quad \left. + \frac{|\varrho_j - \varrho_{j-1}|M_1}{\varsigma_j(1-\varrho_j)(1-\kappa)} + \frac{|\varsigma_j - \varsigma_{j-1}|L}{\varsigma_j(1-\kappa)} \right) \\
&= (1-\delta_j)\|\mu_j - \mu_{j-1}\| + \delta_j \left(\frac{1}{(1-\varrho_j)(1-\kappa)} \frac{\theta_j\|\mu_j - \mu_{j-1}\|}{\varsigma_j} \right. \\
&\quad + \left| 1 - \frac{\varsigma_{j-1}}{\varsigma_j} \right| \frac{L}{(1-\kappa)} + \frac{M_1}{(1-\varrho_j)(1-\kappa)} \frac{|\varrho_j - \varrho_{j-1}|}{\varsigma_j} \\
&\quad \left. + \frac{\varsigma_{j-1}}{\varsigma_j(1-\varrho_j)(1-\kappa)} \frac{\theta_{j-1}\|\mu_{j-1} - \mu_{j-2}\|}{\varsigma_{j-1}} \right),
\end{aligned}
\tag{22}
$$

where $L = \sup_{j\in\mathbb{N}}\{\|\Theta(\nu_{j-1}) - \nu_{j-1}\|\}$, $\delta_j = \varsigma_j(1-\varrho_j)(1-\kappa)$ and $M_1 = \sup_{j\in\mathbb{N}}\{\|\nu_{j-1} - \Upsilon\| + \|\omega_{j-1} - \Upsilon\|\}$. Note that $\frac{\theta_j\|\mu_j - \mu_{j-1}\|}{\varsigma_j} \to 0$ and $\lim_{j\to\infty} \delta_j = 0$. Apply Lemma 2.2 in (22), we get

$$
\|\mu_{j+1} - \mu_j\| \to 0 \quad \text{as} \quad j \to \infty. \tag{23}
$$

Using (18), (23) and triangle inequality, we obtain

$$
\|\mu_{j+1} - \nu_j\| \leq \|\mu_{j+1} - \mu_j\| + \|\mu_j - \nu_j\| \to 0 \quad \text{as} \quad j \to \infty. \tag{24}
$$

From (13) and Lemma 2.5, we have

$$
\|\mu_{j+1} - \Upsilon\|^2 = \|\varrho_j(S(\nu_j) - \Upsilon) + (1-\varrho_j)(R(\omega_j) - \Upsilon)\|^2 \tag{25}
$$

$$
\begin{aligned}
&\leq \varrho_j\|S(\nu_j) - \Upsilon\|^2 + (1-\varrho_j)\|R(\omega_j) - \Upsilon\|^2 - \varrho_j(1-\varrho_j)\|S(\nu_j) - R(\omega_j)\|^2 \\
&\leq \varrho_j\|\nu_j - \Upsilon\|^2 + (1-\varrho_j)\|\omega_j - \Upsilon\|^2 - \varrho_j(1-\varrho_j)\|S(\nu_j) - R(\omega_j)\|^2
\end{aligned}
\tag{26}
$$

and

$$
\begin{aligned}
\|\omega_j - \Upsilon\|^2 &= \|(1-\varsigma_j)(\nu_j - \Upsilon) + \varsigma_j(\Theta(\nu_j) - \Upsilon)\|^2 \\
&\leq (1-\varsigma_j)\|\nu_j - \Upsilon\|^2 + \varsigma_j\|\Theta(\nu_j) - \Upsilon\|^2 \\
&\leq \|\nu_j - \Upsilon\|^2 + \varsigma_j\|\Theta(\nu_j) - \Upsilon\|^2.
\end{aligned}
\tag{27}
$$

Again from (13) and Lemma 2.3, we get

$$
\begin{aligned}
\|\nu_j - \Upsilon\|^2 &= \|(\mu_j - \Upsilon) + \theta_j(\mu_j - \mu_{j-1})\|^2 \\
&\leq \|\mu_j - \Upsilon\|^2 + 2\theta_j\langle \mu_j - \mu_{j-1}, \nu_j - \Upsilon\rangle \\
&\leq \|\mu_j - \Upsilon\|^2 + 2\theta_j\|\mu_j - \mu_{j-1}\|\|\nu_j - \Upsilon\|.
\end{aligned}
\tag{28}
$$

From (25), (27) and (28), we obtain

$$\begin{aligned}\|\mu_{j+1}-\Upsilon\|^2 &\le \varrho_j\|\nu_j-\Upsilon\|^2+(1-\varrho_j)(\|\nu_j-\Upsilon\|^2+\varsigma_j\|\Theta(\nu_j)-\Upsilon\|^2)\\ &\quad -\varrho_j(1-\varrho_j)\|S(\nu_j)-R(\omega_j)\|^2\\ &\le \|\nu_j-\Upsilon\|^2+\varsigma_j(1-\varrho_j)\|\Theta(\nu_j)-\Upsilon\|^2\\ &\quad -\varrho_j(1-\varrho_j)\|S(\nu_j)-R(\omega_j)\|^2\\ &\le \|\mu_j-\Upsilon\|^2+2\theta_j\|\mu_j-\mu_{j-1}\|\|\nu_j-\Upsilon\|\\ &\quad +\varsigma_j(1-\varrho_j)\|\Theta(\nu_j)-\Upsilon\|^2-\varrho_j(1-\varrho_j)\|S(\nu_j)-R(\omega_j)\|^2.\\ \Rightarrow \varrho_j(1-\varrho_j)\|S(\nu_j)-R(\omega_j)\|^2 &\le \|\mu_j-\Upsilon\|^2-\|\mu_{j+1}-\Upsilon\|^2+2\theta_j\|\mu_j-\mu_{j-1}\|\|\nu_j-\Upsilon\|\\ &\quad +\varsigma_j(1-\varrho_j)\|\Theta(\nu_j)-\Upsilon\|^2\\ &\le (\|\mu_j-\Upsilon\|+\|\mu_{j+1}-\Upsilon\|)\|\mu_{j+1}-\mu_j\|\\ &\quad +2\theta_j\|\mu_j-\mu_{j-1}\|\|\nu_j-\Upsilon\|+\varsigma_j(1-\varrho_j)\|\Theta(\nu_j)-\Upsilon\|^2.\end{aligned} \tag{29}$$

Since $\{\mu_j\}$ is bounded, $\lim\limits_{j\to\infty}\|\mu_{j+1}-\mu_j\|=0$, $\lim_{j\to\infty}\varsigma_j=0$ and $0<a\le\varrho_j\le b<1$, therefore from (29), we get

$$\|S(\nu_j)-R(\omega_j)\|\to 0 \quad \text{as} \quad j\to\infty \tag{30}$$

and from (13), we get

$$\|\mu_{j+1}-S(\nu_j)\|=(1-\varrho_j)\|S(\nu_j)-R(\omega_j)\|\le\|S(\nu_j)-R(\omega_j)\|\to 0 \quad \text{as} \quad j\to\infty. \tag{31}$$

Using (24), (31) and triangle inequality, we get

$$\|S(\nu_j)-\nu_j\|\le\|S(\nu_j)-\mu_{j+1}\|+\|\mu_{j+1}-\nu_j\|\to 0 \quad \text{as} \quad j\to\infty. \tag{32}$$

Again from (13) and (30), we get

$$\|\mu_{j+1}-R(\omega_j)\|=\varrho_j\|S(\nu_j)-R(\omega_j)\|\le\|S(\nu_j)-R(\omega_j)\|\to 0 \quad \text{as} \quad j\to\infty$$

and

$$\|R(\omega_j)-\omega_j\|\le\|R(\omega_j)-\mu_{j+1}\|+\|\mu_{j+1}-\omega_j\|\to 0 \quad \text{as} \quad j\to\infty. \tag{33}$$

Let $\{\mu_{j_i}\}$ be a subsequence of $\{\mu_j\}$ such that

$$\limsup_{j\to\infty}\langle\Theta(\Upsilon)-\Upsilon,\mu_j-\Upsilon\rangle=\limsup_{j\to\infty}\langle\Theta(\Upsilon)-\Upsilon,\mu_{j_i}-\Upsilon\rangle. \tag{34}$$

From (18) and (19), we obtain

$$\begin{aligned}&\limsup_{j\to\infty}\langle\Theta(\Upsilon)-\Upsilon,\omega_j-\Upsilon\rangle \\ &\quad=\limsup_{j\to\infty}(\langle\Theta(\Upsilon)-\Upsilon,\omega_j-\nu_j\rangle+\langle\Theta(\Upsilon)-\Upsilon,\nu_j-\mu_j\rangle\\ &\qquad+\langle\Theta(\Upsilon)-\Upsilon,\mu_j-\Upsilon\rangle)\\ &\quad\le\limsup_{j\to\infty}(\|\Theta(\Upsilon)-\Upsilon\|\|\omega_j-\nu_j\|+\|\Theta(\Upsilon)-\Upsilon\|\|\nu_j-\mu_j\|\\ &\qquad+\langle\Theta(\Upsilon)-\Upsilon,\mu_j-\Upsilon\rangle)\end{aligned} \tag{35}$$

$$\begin{aligned}
&\leq \limsup_{j\to\infty} \|\Theta(\Upsilon) - \Upsilon\|\|\omega_j - \nu_j\| + \limsup_{j\to\infty} \|\Theta(\Upsilon) - \Upsilon\|\|\nu_j - \mu_j\| \\
&\quad + \limsup_{j\to\infty} \langle \Theta(\Upsilon) - \Upsilon, \mu_j - \Upsilon\rangle \\
&= \limsup_{j\to\infty} \langle \Theta(\Upsilon) - \Upsilon, \mu_j - \Upsilon\rangle. \qquad (36)
\end{aligned}$$

Since $\{\mu_j\}$ is bounded, we may assume that $\mu_{j_i} \rightharpoonup \mu^*$ for some $\mu^* \in E$.

Using Lemma 2.4, (32) and (33), we have

$$\mu^* \in \Gamma$$

and from (35), we have

$$\begin{aligned}
\limsup_{j\to\infty} \langle \Theta(\Upsilon) - \Upsilon, \omega_j - \Upsilon\rangle &\leq \limsup_{j\to\infty} \langle \Theta(\Upsilon) - \Upsilon, \mu_j - \Upsilon\rangle \\
&= \limsup_{j\to\infty} \langle \Theta(\Upsilon) - \Upsilon, \mu_{j_i} - \Upsilon\rangle \\
&= \langle \Theta(\Upsilon) - \Upsilon, \mu^* - \Upsilon\rangle \leq 0. \qquad (37)
\end{aligned}$$

From (13) and Lemma 2.3, we have

$$\begin{aligned}
\|\omega_j - \Upsilon\|^2 &= \|\varsigma_j\Theta(\nu_j) + (1-\varsigma_j)\nu_j - \Upsilon\|^2 \\
&= \|(1-\varsigma_j)(\nu_j - \Upsilon) + \varsigma_j(\Theta(\nu_j) - \Upsilon)\|^2 \\
&\leq (1-\varsigma_j)^2\|\nu_j - \Upsilon\|^2 + 2\varsigma_j\langle \Theta(\nu_j) - \Upsilon, \omega_j - \Upsilon\rangle. \qquad (38)
\end{aligned}$$

Note that

$$2\|\Theta(\nu_j) - \Theta(\Upsilon)\|\|\omega_j - \Upsilon\| \leq 2\kappa\|\nu_j - \Upsilon\|\|\omega_j - \Upsilon\| \leq \kappa(\|\nu_j - \Upsilon\|^2 + \|\omega_j - \Upsilon\|^2).$$

From (13), (27), (28) and Lemma 2.3, we have

$$\begin{aligned}
\|\mu_{j+1} - \Upsilon\|^2 &= \|\varrho_j S(\nu_j) + (1-\varrho_j)R(\omega_j) - \Upsilon\|^2 \\
&\leq \varrho_j\|S(\nu_j) - \Upsilon\|^2 + (1-\varrho_j)\|R(\omega_j) - \Upsilon)\|^2 \\
&\leq \varrho_j\|\nu_j - \Upsilon\|^2 + (1-\varrho_j)\|\omega_j - \Upsilon\|^2 \\
&\leq \varrho_j\|\nu_j - \Upsilon\|^2 + (1-\varrho_j)\big((1-\varsigma_j)^2\|\nu_j - \Upsilon\|^2 + 2\varsigma_j\langle\Theta(\nu_j) - \Upsilon, \omega_j - \Upsilon\rangle\big) \\
&= [\varrho_j + (1-\varrho_j)(1-\varsigma_j)^2]\|\nu_j - \Upsilon\|^2 + 2(1-\varrho_j)\varsigma_j\langle\Theta(\Upsilon) - \Upsilon, \omega_j - \Upsilon\rangle \\
&\quad + 2(1-\varrho_j)\varsigma_j\langle\Theta(\nu_j) - \Theta(\Upsilon), \omega_j - \Upsilon\rangle \\
&\leq [\varrho_j + (1-\varrho_j)(1-\varsigma_j)^2]\|\nu_j - \Upsilon\|^2 + 2(1-\varrho_j)\varsigma_j\langle\Theta(\Upsilon) - \Upsilon, \omega_j - \Upsilon\rangle \\
&\quad + 2(1-\varrho_j)\varsigma_j\|\Theta(\nu_j) - \Theta(\Upsilon)\|\|\omega_j - \Upsilon\| \\
&\leq [\varrho_j + (1-\varrho_j)(1-\varsigma_j)^2]\|\nu_j - \Upsilon\|^2 + 2\varsigma_j(1-\varrho_j)\langle\Theta(\Upsilon) - \Upsilon, \omega_j - \Upsilon\rangle \\
&\quad + \varsigma_j\kappa(1-\varrho_j)(\|\nu_j - \Upsilon\|^2 + \|\omega_j - \Upsilon\|^2) \\
&= [\varrho_j + (1-\varrho_j)(1-\varsigma_j)^2 + \varsigma_j\kappa(1-\varrho_j)]\|\nu_j - \Upsilon\|^2 \\
&\quad + \varsigma_j\kappa(1-\varrho_j)\|\omega_j - \Upsilon\|^2 + 2\varsigma_j(1-\varrho_j)\langle\Theta(\Upsilon) - \Upsilon, \omega_j - \Upsilon\rangle \\
&\leq [\varrho_j + (1-\varrho_j)(1-\varsigma_j)^2 + 2\varsigma_j\kappa(1-\varrho_j)]\|\nu_j - \Upsilon\|^2
\end{aligned}$$

$$
\begin{aligned}
&+\varsigma_j^2\kappa(1-\varrho_j)\|\Theta(\nu_j)-\Upsilon\|^2+2\varsigma_j(1-\varrho_j)\langle\Theta(\Upsilon)-\Upsilon,\omega_j-\Upsilon\rangle\\
&=[1-2\varsigma_j(1-\varrho_j)(1-\kappa)]\|\nu_j-\Upsilon\|^2+\varsigma_j^2(1-\varrho_j)\|\nu_j-\Upsilon\|^2\\
&\quad+\varsigma_j^2(1-\varrho_j)\kappa\|\Theta(\nu_j)-\Upsilon\|^2+2(1-\varrho_j)\varsigma_j\langle\Theta(\Upsilon)-\Upsilon,\omega_j-\Upsilon\rangle\\
&\le[1-2\varsigma_j(1-\varrho_j)(1-\kappa)](\|\mu_j-\Upsilon\|^2+2\theta_j\|\mu_j-\mu_{j-1}\|\|\nu_j-\Upsilon\|)\\
&\quad+\varsigma_j^2(1-\varrho_j)\|\nu_j-\Upsilon\|^2+\varsigma_j^2(1-\varrho_j)\kappa\|\Theta(\nu_j)-\Upsilon\|^2\\
&+2(1-\varrho_j)\varsigma_j\langle\Theta(\Upsilon)-\Upsilon,\omega_j-\Upsilon\rangle\\
&\le[1-2\varsigma_j(1-\varrho_j)(1-\kappa)]\|\mu_j-\Upsilon\|^2+2\theta_j\|\mu_j-\mu_{j-1}\|\|\nu_j-\Upsilon\|\\
&\quad+\varsigma_j^2(1-\varrho_j)\kappa\|\Theta(\nu_j)-\Upsilon\|^2+\varsigma_j^2(1-\varrho_j)\|\nu_j-\Upsilon\|^2\\
&\quad+2(1-\varrho_j)\varsigma_j\langle\Theta(\Upsilon)-\Upsilon,\omega_j-\Upsilon\rangle\\
&=(1-\gamma_j)\|\mu_j-\Upsilon\|^2+\gamma_j\Bigg(\frac{\theta_j}{\varsigma_j(1-\varrho_j)(1-\kappa)}\|\mu_j-\mu_{j-1}\|\|\nu_j-\Upsilon\|\\
&\quad+\frac{\varsigma_j k}{2(1-\kappa)}\|\Theta(\nu_j)-\Upsilon\|^2+\frac{\varsigma_j}{2(1-\kappa)}\|\nu_j-\Upsilon\|^2+\frac{1}{1-\kappa}\langle\Theta(\Upsilon)-\Upsilon,\omega_j-\Upsilon\rangle\Bigg)\\
&\le(1-\gamma_j)\|\mu_j-\Upsilon\|^2+\gamma_j\Bigg(\frac{\theta_j}{\varsigma_j(1-\varrho_j)(1-\kappa)}\|\mu_j-\mu_{j-1}\|M_2\\
&\quad+\frac{\varsigma_j k}{2(1-\kappa)}M_3+\frac{\varsigma_j}{2(1-\kappa)}M_2^2+\frac{1}{1-\kappa}\langle\Theta(\Upsilon)-\Upsilon,\omega_j-\Upsilon\rangle\Bigg)\\
&\le(1-\gamma_j)\|\mu_j-\Upsilon\|^2+\gamma_j\eta_j, \qquad (39)
\end{aligned}
$$

where $\gamma_j=2\varsigma_j(1-\varrho_j)(1-\kappa)$, $M_2=\sup_j\{\|\nu_j-\Upsilon\|\}$, $M_3=\sup_j\{\|\Theta(\nu_j)-\Upsilon\|\}$ and

$$
\eta_j=\frac{\theta_j M_2}{\varsigma_j(1-\varrho_j)(1-\kappa)}\|\mu_j-\mu_{j-1}\|+\frac{\varsigma_j\kappa M_3}{2(1-\kappa)}+\frac{\varsigma_j M_2^2}{2(1-\kappa)}+\frac{\langle\Theta(\Upsilon)-\Upsilon,\omega_j-\Upsilon\rangle}{1-\kappa}.
$$

Note that $\lim_{j\to\infty}\gamma_j=0$, $\sum_{j=1}^{\infty}\gamma_j=\infty$ and from (19), (37), we get

$$
\limsup_{j\to\infty}\eta_j\le 0.
$$

Therefore, using the Lemma 2.2, we get

$$
\mu_j\to\Upsilon.
$$

The proof is now complete. ■

If we take $S=R$ and $\theta_j=0$ for all $j\in\mathbb{N}$ and $\Theta(\mu)=u$ for all $\mu\in E$, then Theorem 3.1 reduces to the result [31, Theorem 5.4].

Corollary 3.2 ([31, Theorem 5.4]): *Let E be a nonempty CCS of $\mathcal{H}$ and let $R:E\to E$ be an NE mapping such that $Fix(R)\neq\emptyset$. Assume that $\{\varsigma_j\}$ and $\{\varrho_j\}$ be real sequences which fulfils the requirements $(\curlywedge_1)$ and $(\curlywedge_2)$. Then the sequence $\{\mu_j\}$ created by*

$$
\begin{cases}\omega_j=\varsigma_j u+(1-\varsigma_j)\mu_j,\\ \mu_{j+1}=\varrho_j R(\mu_j)+(1-\varrho_j)R(\omega_j), \quad j\in\mathbb{N},\ u\in E\end{cases}
$$

converges strongly to $\mu^*=P_{Fix(R)}(u)$.

Proof: From Theorem 3.1, the proof of the corollary can be obtained by accepting $\Theta(\mu) = u$ for every $\mu \in E$ and $S = R$ and $\theta_j = 0$ for all $j \in \mathbb{N}$. ■

Remark 3.3: Our Algorithm 3.1 provides strong convergence results in real Hilbert spaces, while Sahu et al. [33] have proved the weak convergence results. Our result is improvement over the results of Sahu [31, Theorem 5.4] and Qin et al. [29, Theorem 2.1].

3.1. CS for variational inequality problems

Let E be a nonempty closed affine subset of $\mathcal{H}$. Let $\alpha > 0$ be constant, $A : E \to \mathcal{H}$ be a α-ISM mapping and P_E be the metric projection mapping from $\mathcal{H}$ onto E. For $\lambda \in (0, 2\alpha)$ the mapping $P_E(I - \lambda A)$ is NE operator [31].

Problem 3.4: Let E be a nonempty closed affine subset of $\mathcal{H}$ and let $A, B : E \to \mathcal{H}$ be two α-ISM and β-ISM mappings, respectively. The objective is to find $\mu^* \in E$ such that

$$\mu^* \in \Omega = \Omega_A \cap \Omega_B,$$

where $\Omega_A = \{\mu^* \in E : \langle A\mu^*, \mu - \mu^*\rangle \geq 0, \forall \mu \in E\}$ and $\Omega_B = \{\mu^* \in E : \langle B\mu^*, \mu - \mu^*\rangle \geq 0, \forall \mu \in E\}$.

We now give an application of Theorem 3.1 for finding solutions to Problem 3.4.

Theorem 3.5: *Let E be a nonempty closed affine subset of $\mathcal{H}$. Let $\alpha, \beta > 0$, $A, B : E \to \mathcal{H}$ be two α-ISM and β-ISM mappings respectively, such that the solution set $\Omega \neq \emptyset$. Assume that $\Theta : E \to E$ be a contraction mapping and $\{\varsigma_j\}$, $\{\varrho_j\}$, $\{\theta_j\}$ are sequences in $(0, 1)$ which fulfils the requirements $(\curlywedge_1)$, $(\curlywedge_2)$ and $(\curlywedge_3)$. Let $\gamma_1 \in (0, 2\alpha)$ and $\gamma_2 \in (0, 2\beta)$. Let $\{\mu_j\}$ be a sequence in E generated from $\mu_0, \mu_1 \in E$ and defined by*

$$\begin{cases} \nu_j = \mu_j + \theta_j(\mu_j - \mu_{j-1}); \\ \omega_j = \varsigma_j\Theta(\nu_j) + (1 - \varsigma_j)\nu_j; \\ \mu_{j+1} = \varrho_j S(\nu_j) + (1 - \varrho_j)R(\omega_j), \quad j \in \mathbb{N} \end{cases} \tag{40}$$

where $S, R : E \to E$ defined by $S\mu = P_E(\mu - \gamma_1 A\mu)$ and $R\mu = P_E(\mu - \gamma_1 B\mu)$ for all $\mu \in E$. Then the sequence $\{\mu_j\}$ converges strongly to the unique solution $\mu^* \in \Omega$ of the VIP

$$\langle (I - \Theta)\mu^*, \mu - \mu^*\rangle \geq 0 \quad \text{for all } \mu \in \Omega.$$

Proof: From the proof of Theorem 3.1, the theorem's proof is derived. ■

4. Numerical examples

This section offers a few numerical examples that demonstrate the suggested algorithm's efficiency and convergence. We will compare our Algorithm 3.1 with the iteration process (10) of Sahu [31] and (11) of Qin et al. [29] to illustrate the performance of Algorithm 3.1. For Algorithm 3.1, we take the contraction mapping $\Theta : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$\Theta(\mu_1, \mu_2, \mu_3) = \left(\frac{\mu_1}{2}, \frac{\mu_2}{3}, \frac{\mu_3}{4}\right) \quad \text{for all } (\mu_1, \mu_2, \mu_3) \in \mathbb{R}^3.$$

For the comparison of our Algorithm 3.1 with iteration processes (10) and (11), we take $\mu_0 = (0.1, 0.3, 0.1)$, $\mu_1 = (0.2, 1.0, 0.1)$, $u = (0.01, 0.02, 0.1)$ and $\theta_j = 0.3$, $\varsigma_j = \frac{1}{\sqrt{j+2}}$, $\varrho_j = 0.4$ for each $j \in \mathbb{N}$. We use the condition $E_j = \|\mu_{j+1} - \mu_j\| \leq 10^{-6}$ to terminate the all iteration processes.

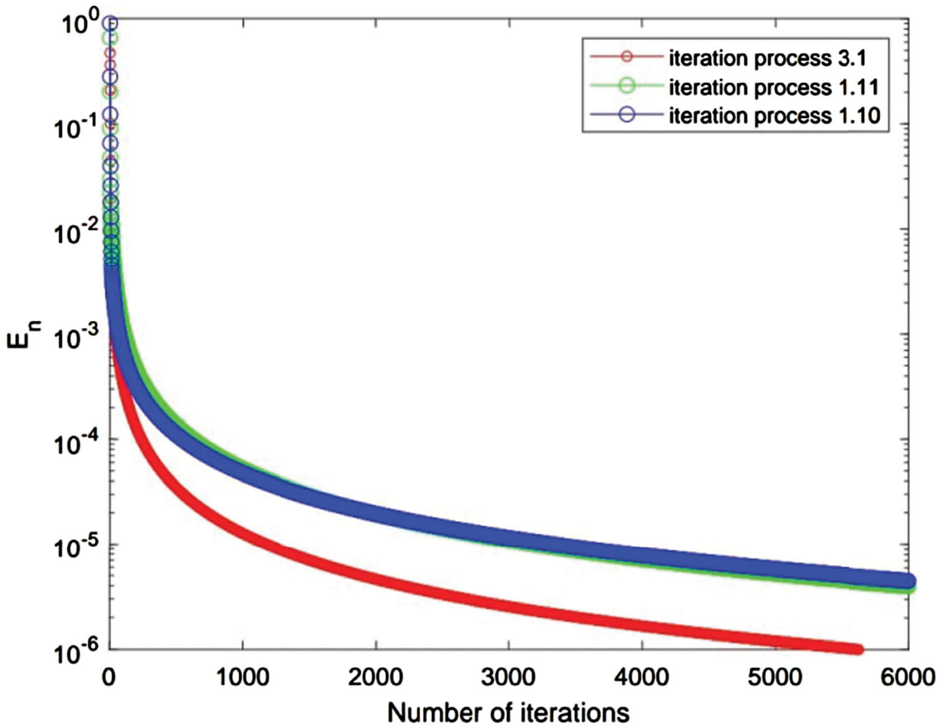

Figure 1. Comparison of Algorithm 3.1 with iteration processes (10) and (11).

Example 4.1: Let $A = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix}$, $b = (3, 0, 1)$ and $d = (2, 0, 3)$. Let $\mathcal{H} = \mathbb{R}^3$ with usual norm and $g, h : \mathbb{R}^3 \to \mathbb{R}$ be the two mappings defined by

$$g(\mu) = \frac{1}{2}\|A\mu - b\|^2 \quad \text{and} \quad h(\mu) = \frac{1}{2}\|B\mu - d\|^2 \quad \text{for all } \mu \in \mathbb{R}^3.$$

Note that $\nabla g(\mu) = A^*(A\mu - b)$ for all $\mu \in \mathbb{R}^3$, it follows that

$$\|\nabla g(\mu) - \nabla g(\nu)\| = \|A^*(A\mu - b) - A^*(A\nu - b)\| \leq \|A\|^2\|\mu - \nu\|$$

for all $\mu, \nu \in \mathbb{R}^3$. Hence, $\nabla g(\mu)$ and $\nabla h(\mu)$ are $\frac{1}{\|A\|^2}$ and $\frac{1}{\|B\|^2}$-ISM operators respectively [5], and hence $(I - \gamma_1 \nabla g)$ and $(I - \gamma_2 \nabla h)$ are NE from $\mathbb{R}^3$ into $\mathbb{R}^3$ for $\gamma_1 \in (0, \frac{2}{\|A\|^2})$ and $\gamma_2 \in (0, \frac{2}{\|B\|^2})$. Here, $\|A\| = 5.7093$ and $\|B\| = 5$. Therefore, $R, S : \mathbb{R}^3 \to \mathbb{R}^3$ be the NE mappings defined by

$$S(\mu) = \mu - \gamma_1 A^*(A\mu - b) \quad \text{and} \quad R(\mu) = \mu - \gamma_2 B^*(B\mu - d)$$

for $\gamma_1 \in (0, \frac{2}{32.5958})$ and $\gamma_2 \in (0, \frac{2}{25})$. Take $E = \mathbb{R}^3$ the VIPs have a CS $\mu^* = (1, 0, 0) \in \Omega_{\nabla g} \cap \Omega_{\nabla h}$. By using the Theorem 3.5, we can find the solution of Problem 3.4. The numerical results are presented in Figure 1.

Example 4.2: Let $\mathcal{H} = \mathbb{R}^3$ with usual norm and $S, R : \mathbb{R}^3 \to \mathbb{R}^3$ be the mappings defined by

$$R(\mu_1, \mu_2, \mu_3) = \left(\sin\frac{\mu_1 + \mu_2}{2}, \sin\frac{\mu_1 - \mu_2}{2}, \frac{\sin 2\mu_3}{2}\right) \quad \text{for all} \quad (\mu_1, \mu_2, \mu_3) \in \mathbb{R}^3$$

and

$$S(\mu_1,\mu_2,\mu_3)=\left(\frac{\mu_1+\sin\mu_1}{2},\frac{\mu_2+\sin\mu_2}{2},\frac{\mu_3+\sin\mu_3}{2}\right)\quad \text{for all } (\mu_1,\mu_2,\mu_3)\in\mathbb{R}^3$$

both S, R are NE mapping with $(0,0,0)$ is a common FP of S and R. Let $\mu=(\mu_1,\mu_2,\mu_3)$, $\nu=(\nu_1,\nu_2,\nu_3)\in\mathbb{R}^3$ be any two points, then

$$\begin{aligned}\|R(\mu)-R(\nu)\|^2&=\left\|\left(\sin\frac{\mu_1+\mu_2}{2},\sin\frac{\mu_1-\mu_2}{2},\frac{\sin 2\mu_3}{2}\right)-\left(\sin\frac{\nu_1+\nu_2}{2},\sin\frac{\nu_1-\nu_2}{2},\frac{\sin 2\nu_3}{2}\right)\right\|^2\\&\le\left|\frac{\mu_1+\mu_2}{2}-\frac{\nu_1+\nu_2}{2}\right|^2+\left|\frac{\mu_1-\mu_2}{2}-\frac{\nu_1-\nu_2}{2}\right|^2+\frac{1}{4}|2\mu_3-2\nu_3|^2\\&\le 2\left|\frac{\mu_1-\nu_1}{2}\right|^2+2\left|\frac{\mu_2-\nu_2}{2}\right|^2+2\left|\frac{\mu_1-\nu_1}{2}\right|^2+2\left|\frac{\mu_2-\nu_2}{2}\right|^2+|\mu_3-\nu_3|^2\\&=|\mu_1-\nu_1|^2+|\mu_2-\nu_2|^2+|\mu_3-\nu_3|^2=\|\mu-\nu\|^2\end{aligned}$$

which implies

$$\|R(\mu)-R(\nu)\|\le\|\mu-\nu\|\quad\forall\,\mu,\nu\in\mathbb{R}^3.$$

Similarly

$$\begin{aligned}\|S(\mu)-S(\nu)\|^2&=\left\|\left(\frac{\mu_1+\sin\mu_1}{2},\frac{\mu_2+\sin\mu_2}{2},\frac{\mu_3+\sin\mu_3}{2}\right)\right.\\&\quad\left.-\left(\frac{\nu_1+\sin\nu_1}{2},\frac{\nu_2+\sin\nu_2}{2},\frac{\nu_3+\sin\nu_3}{2}\right)\right\|^2\\&=\left|\frac{\mu_1+\sin\mu_1}{2}-\frac{\nu_1+\sin\nu_1}{2}\right|^2+\left|\frac{\mu_2+\sin\mu_2}{2}-\frac{\nu_2+\sin\nu_2}{2}\right|^2\Bigg|\\&\quad+\left|\frac{\mu_3+\sin\mu_3}{2}-\frac{\nu_3+\sin\nu_3}{2}\right|^2\\&\le\frac{1}{4}\left(2|\mu_1-\nu_1|^2+2|\sin\mu_1-\sin\nu_1|^2+2|\mu_2-\nu_2|^2+2|\sin\mu_2-\sin\nu_2|^2\right.\\&\quad\left.+2|\mu_3-\nu_3|^2+|\sin\mu_3-\sin\nu_3|^2\right)\\&\le\frac{1}{4}\left(2|\mu_1-\nu_1|^2+2|\mu_1-\nu_1|^2+2|\mu_2-\nu_2|^2+2|\mu_2-\nu_2|^2\right.\\&\quad\left.+2|\mu_3-\nu_3|^2+|\mu_3-\nu_3|^2\right)\\&\le|\mu_1-\nu_1|^2+|\mu_2-\nu_2|^2+|\mu_3-\nu_3|^2=\|\mu-\nu\|^2\end{aligned}$$

which implies

$$\|S(\mu)-S(\nu)\|\le\|\mu-\nu\|\quad \textit{forall}\ \mu,\nu\in\mathbb{R}^3.$$

Hence both S and R are NE operators. The numerical results are presented in Figure 2.

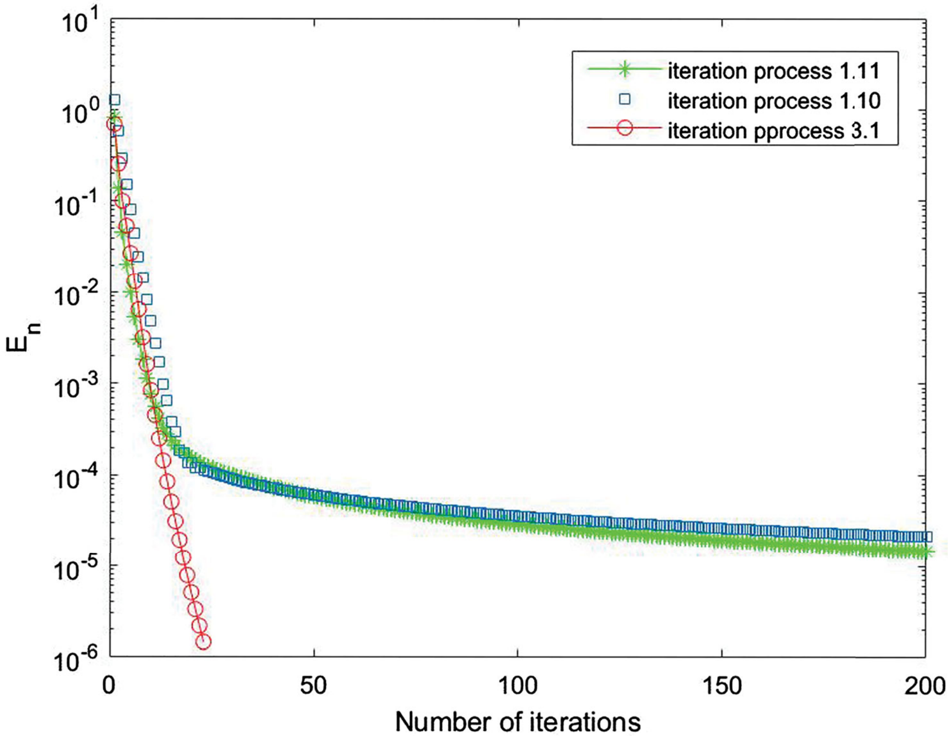

Figure 2. Comparison of Algorithm 3.1 with iteration processes (10) and (11).

Example 4.3: Let $\mathcal{H} = \mathbb{R}^2$ with usual norm and $S, R : \mathbb{R}^2 \to \mathbb{R}^2$ be the mappings defined by

$$S(\mu_1, \mu_2) = \left(\frac{\mu_1 + \sin \mu_1}{2}, \frac{\mu_2 + \sin \mu_2}{2} \right) \quad \text{for all } (\mu_1, \mu_2) \in \mathbb{R}^2$$

and

$$R(\mu_1, \mu_2) = \left(\sin \frac{\mu_1 + \mu_2}{2}, \sin \frac{\mu_1 - \mu_2}{2} \right) \quad \text{for all } (\mu_1, \mu_2) \in \mathbb{R}^2$$

both S and R are NE mapping possessing a common FP $(0, 0)$.

Let $\Theta : \mathcal{H} \to \mathcal{H}$ be a contraction mapping defined by

$$\Theta(\mu_1, \mu_2) = \left(\frac{\mu_1}{2}, \frac{\mu_2}{3} \right) \quad \text{for all } (\mu_1, \mu_2) \in \mathbb{R}^2.$$

Take $\mu_0 = (0.1, 0.3)$, $\mu_1 = (0.2, 1.0)$, $\theta_j = \frac{1}{3}$, $\varsigma_j = \frac{1}{\sqrt{j+1}}$ and different values of $\varrho_j = 0.2, 0.4, 0.6, 0.8$ in Algorithm 3.1, iteration process (10) and (11). The numerical results are described in Table 1 and for $\varrho_j = 0.4$ in Figure 3. The results demonstrate that Algorithm 3.1 performs better than the iteration process (10) and (11).

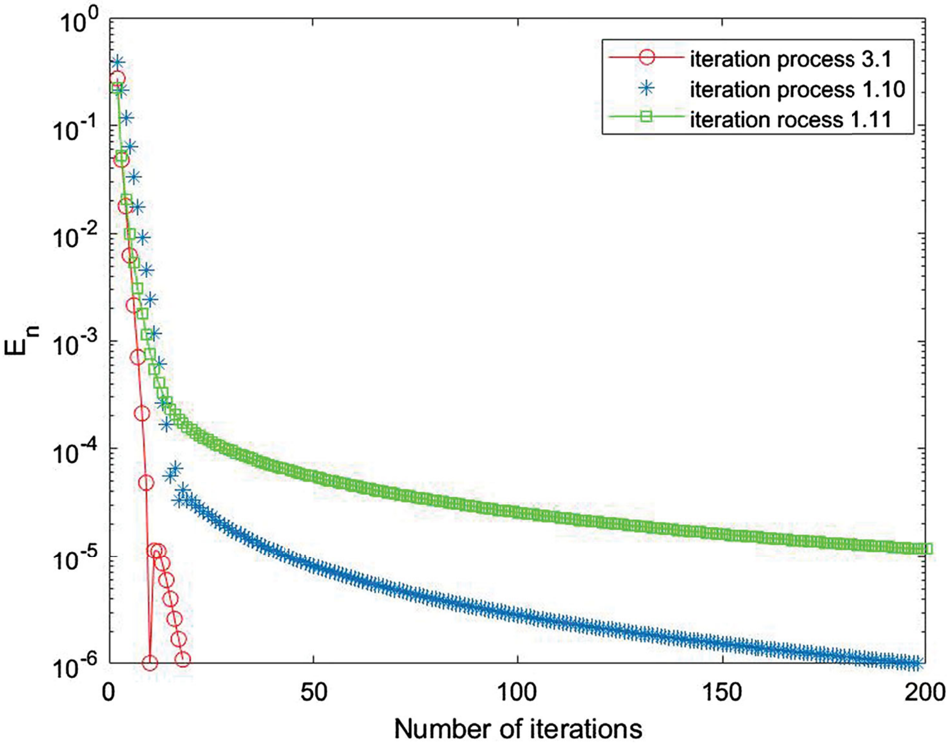

Figure 3. Comparison of Algorithm 3.1 with iteration processes (10) and (11).

Table 1. Comparison of Algorithm 3.1 with iteration processes (10) and (11) by elapsed time in seconds.

No. of iterations	ϱ_j	Algorithm 3.1	Iteration process 10	Iteration process 11
100	0.2	0.006783	0.006963	0.006838
100	0.4	0.006619	0.006896	0.006851
100	0.6	0.005909	0.006997	0.007004
100	0.8	0.006393	0.006935	0.006813

5. Conclusion

We presented a novel inertial viscosity approximation algorithm in a real Hilbert space in this article to examine a CE of the set of FPs of two nonexpansive operators. Our findings were then used to determine the CS of two variational inequality problems, and the suggested approach showed strong convergence. The major contribution of our proposed Algorithm 3.1 is to find the common solution of a pair of VIPs and fast convergence in comparison to the iteration processes (10) and (11) having the same number of iterations. Also the importance of our Algorithm has been mentioned in Remark 3.3 in relation to existing results in the current literature. Lastly, we provided a few numerical examples to show how effective our suggested methodology is compared to some other iterative techniques.

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Disclosure statement

No potential conflict of interest was reported by the author(s).

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